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Dunkl operator and matrix models at finite N

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Table of contents

Introduction

Outline

Correlation functions in Gaussian matrix model

The partition function for the $\beta = \frac{1}{\alpha}$-deformed Gaussian eigenvalue model can be written in eigenvalue form

$$Z \stackrel{\text{def}}{=} \int \left(\prod_{i=1}^{N} dx_i \right) w_\beta(x) \exp\left[-\sum_{i=1}^{N} \frac{x_i^2}{2} \right]. \tag{1}$$

where $w_\beta(x) = \prod_{1 \le i < j \le N} (x_i - x_j)^{2\beta}$ is the Vandermonde and x_i are diagonalized elements of Hermitian matrix $H = h_{ij}$. Our goal is to find a multipoint correlation of the gaussian model

$$\mathbb{E}\left[Tr(H^{k_1}) \dots Tr(H^{k_l}) \right] = \mathbb{E}\left[\left(\sum_{i=1}^{N} x_i^{k_1} \right) \dots \left(\sum_{i=1}^{N} x_i^{k_l} \right) \right] \tag{2}$$

We add generating function with parameter q_k to the partition function:

$$Z(q_k) = \int \left(\prod_{i=1}^{N} dx_i \right) w_\beta(x) \exp\left[-\sum_{i=1}^{N} \frac{x_i^2}{2} + x_i \right] \exp\left[\beta \sum_{k=1}^{\infty} \frac{q_k}{k} \sum_{i=1}^{N} x_i^k \right]. \tag{3}$$

Connection to supersymmetric theory

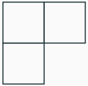
In 3d $\mathcal{N} = 2$ supersymmetry on $S_1 \times_q D_2$, consider a vector multiplet, an adjoint chiral multiplet of mass t, and two antichiral multiplet of mass u_1, u_2. Then the partition function of the theory can be computed using supersymmetric localization. The integrant is

$$Z(\lambda) = \prod_{1 \leq k \neq l \leq N} \frac{(\lambda_k/\lambda_l; q)_\infty}{(t\lambda_k/\lambda_l; q)_\infty} \prod_{j=1}^{N} \prod_{k=1}^{N_f} (q\lambda_j u_k; q)_\infty \ , \qquad (4)$$

After setting $q = t^\beta, \quad \beta \in \mathbb{Z}$ and sending $q \to 1$ we recover the β-model as a semiclassical limit.

Ring of Symmetric polynomials

Symmetric polynomials is a polynomial with n variable such that it remains invariant under permutation of variables.
These polynomials with multiplication form a ring of symmetric polynomials.
Partition of integer n is a way of writing n as a sum of integers. For instance partitions of 3 are $\{\{3\},\{2,1\},\{1,1,1\}\}$.

Young diagram is a set of boxes with left justified rows. Young diagrams can be use for visual representation of partitions of number n.
$\{3\} =$ , $\{2,1\} =$ , $\{1,1,1\} =$ 

(semi-)standard Young tableau is labeling a boxes of a Young diagram (weakly) increasing to the right and increasing downward.

Symmetric function I:

There are many symmetric polynomials.

Monomial symmetric functions:

$$\mathbf{m}_\lambda = \sum_{\mu \sim \lambda} x^\mu, \tag{5}$$

where $\mu \sim \lambda$ means rearrangement of parts of λ.

Powersum symmetric functions:

$$p_\lambda = \prod_{i \in \ell(\lambda)} p_{\lambda_i} \quad \text{and,} \quad p_k = \sum_{i=1}^{n} x_i^k, \tag{6}$$

where $\ell(\lambda)$ is the length of partition λ and λ_i its i-part. We equipped our ring with an inner product $\langle \bullet, \bullet \rangle$

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}, \qquad z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!, \tag{7}$$

m_i is the number the parts in *lambda* equals to i.

Symmetric functions II

Schur polynomial of n variable for a partition λ is

$$S_\lambda(x_1, x_2, \ldots, x_n) = \sum_T \prod_{s \in \lambda} x^T = \sum_T x_1^{t_1} \ldots x_n^{t_n}, \tag{8}$$

where summation is over all semistandard Young tableaux T of shape λ, and t_i is the weight of i in T. It counts the number of occurrence of i in T. As an example we calculate $T(\{2\})=\{ \boxed{1\,|\,1}, \boxed{1\,|\,2}, \boxed{2\,|\,2} \}$.
So we have

$$S_{\{2\}} = x_1^2 + x_1 x_2 + x_2^2 = \frac{p_1^2}{2} + \frac{p_2}{2}. \tag{9}$$

In the last equality we use the fact that power sum and Schur polynomials form a complete orthonormal basis for the space of symmetric functions

$$\langle S_\lambda, S_\mu \rangle = \delta_{\lambda,\mu}. \tag{10}$$

Deformation of Hall inner product and Jack polynomials:

We can deform the Hall inner product

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda,\mu} \alpha^{\ell(\lambda)} z_\lambda, \tag{11}$$

with $\alpha \in \mathbb{R}$. This deformation allow us to have a deformation of Schur functions with respect to this new inner product.
Jack polynomials defines uniquely by the fillowing conditions:

$$\langle P_\lambda, P_\mu \rangle_\alpha = 0, \text{ if } \lambda \neq \mu,$$
$$P_\lambda = \sum_{\mu \leq_d \lambda} C_{\lambda\mu} \mathbf{m}_\mu$$
$$[\mathbf{m}_\lambda] P_\lambda = 1, \quad \text{P-normalization.}$$
$$[\mathbf{m}_\lambda] J_\lambda = |\lambda|!, \quad \text{J-normalization.}$$

Some examples of Jack polynomials

Jack polynomials are one parameter $\beta := \frac{1}{\alpha}$ deformation of Schur polynomials and form orthogonal bases for the space of symmetric polynimals. Here is Jack for integer partition of 3

$$P_{\{3\}} = \frac{\beta^2 p_1^3 + 3\beta p_2 p_1 + 2p_3}{\beta^2 + 3\beta + 2}, \tag{12}$$

$$P_{\{2,1\}} = -\frac{-\beta p_1^3 + (\beta - 1) p_2 p_1 + p_3}{2\beta + 1}, \tag{13}$$

$$P_{\{1,1,1\}} = \frac{1}{6}\left(p_1^3 - 3p_2 p_1 + 2p_3\right). \tag{14}$$

We write $P_{\{2\}}$ to compare the result with example we directly calculate for Schur, when $\beta = 1$

$$P_{\{2\}} = \frac{2\beta x_2 x_1}{\beta + 1} + x_1^2 + x_2^2 = \frac{\beta p_1^2 + p_2}{\beta + 1}. \tag{15}$$

Connection of symmetric polynomials and correlation functions

As we have already seen correlation function can be seen as a symmetric polynomials labeled by a partition for instance

$$\mathbb{E}\left[Tr(H^2)Tr(H^2)\right] = \mathbb{E}\left[(\sum_{i=1}^{N} x_i^2)^2\right] = \mathbb{E}\left[p_2^2\right] = \langle p_{\{2,2\}}\rangle = \mathbb{E}\left[p_{\{2^2\}}\right] \tag{16}$$

Using the fact that Jack polynomials are orthonormal bases we can expand powersum polynomials in these bases

$$p_\lambda = \sum_{\nu \vdash |\lambda|} R_\nu^\lambda P_\nu \tag{17}$$

where R_ν^λ can be computed algebraically, as an example

$$\begin{aligned} p_{\{22\}} =& P_{\{4\}} - \frac{4}{3\alpha+1} P_{\{31\}} - \frac{4\left(\alpha^2+\alpha+1\right)}{(\alpha+1)(2\alpha+1)} P_{\{22\}} \\ &- \frac{4\alpha}{(\alpha+1)^2} P_{\{211\}} + \frac{24\alpha}{\alpha^3+6\alpha^2+11\alpha+6} P_{\{1111\}}. \end{aligned} \tag{18}$$

Question Can we simplify the result of correlation function using other bases? Surprisingly, Yes!

Dunkl Operator

Dunkl Operator

Dunkl operator defines on using a certain data:
Definition:

- Let R be a root system.
- Let G be a reflection group on $R^\vee$
- $k : R \to \mathbb{C}$ a G-invariant function.
- $\sigma_\alpha X := X - \frac{2\langle X,\alpha\rangle}{\langle \alpha,\alpha\rangle}$ is a reflection aloung the root $\alpha \in R$.

Then the Dunkl operator for $\xi \in \mathbb{R}^N$ is

$$T_\xi f(x) := \partial_\xi + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}. \tag{19}$$

We set $T_{\xi_i} = T_i$ for $\xi_i \in \mathbb{R}^N$.

Why is Dunkl operator interesting?

Crucial theorem: Dunkl operators commutes witheach others [C.Dunkl]

$$[T_i, T_j] = 0. \tag{20}$$

Corollary(1): Set $\Pi = \mathbb{C}[\mathbb{R}^N]$ to be a $\mathbb{C}$-algebra of polynomials. Clearly this is a graded space $\Pi = \bigoplus_{n\geq 0} p_n$.
There is an algebra homomorphism $\varphi : \Pi \to \text{End}\ \Pi$ such that:

$$\varphi_k : x_i \mapsto T_i, \quad 1 \mapsto \text{id}. \tag{21}$$

Therefore we can define $p(T) := \varphi_k P$.
Corollary(2): There exist a symmetric non-degenerate inner product on Pi called generalized Fischer product

$$[p, q]_k := (p(T)q)(0). \tag{22}$$

Now one can do harmonic analysis on Dunkl operators.

Dunkl Kernel and Bessel function

Definition/Theorem Dunkl Kernel is $f = E_k(\bullet, y)$ such that

$$T_\xi f = \langle \xi, y \rangle f, \qquad f(0) = 1. \tag{23}$$

It is a unique and real analytic solution for $Re(k > 0)$ [Dunkl].

Some properties:

- $E_k(y, x) = E_k(x, y)$.
- $E_k(gx, gy) = E(x, y)$, for $g \in G$.

And the convolution/Fourier theorem [Dunkl]

$$\int_N E_k(x, y) E_k(x, z) e^{-|x|^2/2} w_k(x) dx = c_k e^{(y,y)/2+(z,z)/2} E_k(z, y).$$

Generalized Bessel function and spherical part of algebra

The following two theorems by [Opdam] and [Okounkov] connect the theory of Dunkl operators to matrix models.

Definition: Generalized bessel function is defined as

$$F_k(x, y) := \frac{1}{|G|} \sum_{f \in G} E_k(gx, y). \tag{24}$$

Theorem: [Okounkov]

$$F_{\frac{1}{\alpha}}(x, y) = \sum_{\lambda} \frac{P_\lambda(x; \alpha) P_\lambda(y; \alpha)}{(n/\alpha)_\lambda \mathfrak{p}_\lambda}. \tag{25}$$

where $(u)_\lambda = \prod_{(i,j) \in \lambda}(u + (j-1) - (i-1)/\alpha)$, and $\mathfrak{p}_\lambda = (P_\lambda, P_\lambda)$ and $Re(\alpha) > 0$.

Result:

Using Okounkov Bessel function formula and Dunkl convolution theorem we can evaluate the average of two Jack polynomials

Theorem:

$$\mathbb{E}[P_\mu(x;\alpha)P_\beta(x;\alpha)] = \alpha^{-|\mu|-|\beta|}J_\mu(1;\alpha)J_\beta(1;\alpha)\times \left\langle P_\mu(y;\alpha), \left\langle P_\beta(z;\alpha), e^{|y|^2/2+|z|^2/2}\sum_\lambda \frac{P_\lambda(y;\alpha)P_\lambda(z;\alpha)}{(n/\alpha)_\lambda \mathfrak{p}_\lambda}\right\rangle\right\rangle \tag{26}$$

Corollary:

$$\mathbb{E}[P_\mu(x;\alpha)e^{-p_1(x)}] = \alpha^{-|\mu|}J_\mu(1;\alpha)e^{\frac{-N}{2}}\left[p_2^{|\mu|/2} - p_1^{|\mu|}\right]P_\mu \tag{27}$$

Thanks.

Cauchy identity

Cauchy identity for Jack polynomials is

$$\exp\left[\beta\sum_{k=1}^{\infty}\frac{p_k\bar{p}_k}{k}\right]=\sum_{\lambda}\frac{P_\lambda\{p_k\}P_\lambda\{\bar{p}_k\}}{\langle P_\lambda,P_\lambda\rangle}. \tag{28}$$

here p_k and $\bar{p}_k$ are power sum polynomials of possibly different sets if variables. Putting $\bar{p}_k = a_2\beta^{-1}\delta_{k,2}$ we see that

$$\exp\left[\frac{a_2 p_2}{2}\right]=\sum_{\lambda}\frac{P_\lambda\cdot P_\lambda\{a_2\beta^{-1}\delta_{k,2}\}}{\langle P_\lambda,P_\lambda\rangle}. \tag{29}$$

But the left hand side can also obtain by mutiplying p_2 n-times on $P_\varnothing$. This gives us the expression for C's

$$\sum_{\lambda^{(2)}\ldots\lambda^{(2n-2)};\lambda^{(2n)}=\lambda} C_{\phi\lambda^{(2)}}C_{\lambda^{(2)}\lambda^{(4)}}\ldots C_{\lambda^{(2n-2)}\lambda^{(2n)}}=\frac{2^{|\lambda|/2}(|\lambda|/2)!}{a_2^{|\lambda|/2}}\frac{P_\lambda\{a_2\beta^{-1}\delta_{k,2}\}}{\langle P_\lambda,P_\lambda\rangle}. \tag{30}$$

A and Peri rule

First, we notice that

$$A_{\mu\lambda} = \frac{\langle P_\lambda, p_1^2 P_\mu \rangle}{\langle P_\lambda, p_2 P_\mu \rangle} = \frac{1}{1 - 2\frac{\langle P_\lambda, P_{(1,1)} P_\mu \rangle}{\langle P_\lambda, P_1^2 P_\mu \rangle}}. \tag{31}$$

The last equality follows from $P_{\{1\}} = p_1$ and $P_{\{1,1\}} = \frac{1}{2}\left(p_1^2 - p_2\right)$. Jack polynomials are satisfying a rule known as Pieri rule

$$P_{(1^r)} P_\mu = \sum_\lambda c^\lambda_{\mu,(1^r)} P_\lambda, \tag{32}$$

where $\lambda - \mu$ is vertical r-strip and $c^\lambda_{\mu,(1^r)}$ is known object. Using this rule we can calculate $A_{\mu\nu}$

$$\beta(1-\beta)A_{\mu\lambda} = (j_2 - j_1 + \beta(i_1 - i_2))^2 - (1 - \beta + \beta^2) \tag{33}$$

Peri rule coefficients

The coefficients in Peri rule 32 is given by

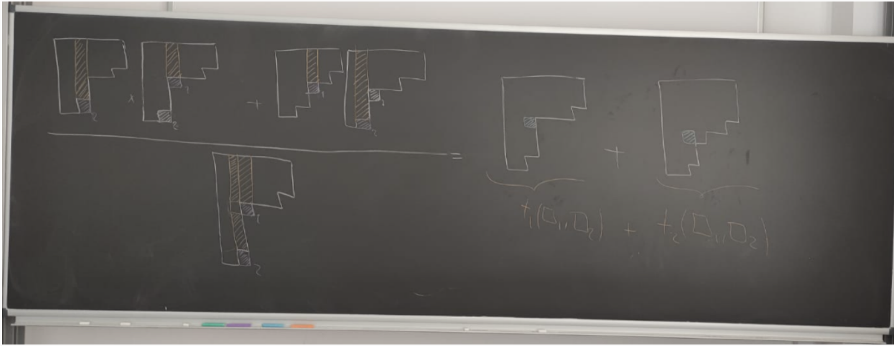
$$c^{\lambda}_{\mu,(1^r)} = \prod_{s \in X(\lambda/\mu)} \frac{h^{\lambda}_{*}(s) h^{*}_{\mu}(s)}{h^{\mu}_{*}(s) h^{*}_{\lambda}(s)} \tag{34}$$

$h^{*}_{\lambda}(s)$ and $h^{\lambda}_{*}(s)$ are respectively the upper and lower hook lengths of the box s:

$$\begin{aligned} h^{*}_{\lambda}(s) &= \beta^{-1}(a_{\lambda}(s)+1) + l_{\lambda}(s) \\ h^{\lambda}_{*}(s) &= \beta^{-1} a_{\lambda}(s) + l_{\lambda}(s) + 1 \end{aligned} \tag{35}$$

$$\frac{\langle P_{\lambda}, P_1^2 P_{\mu} \rangle}{\langle P_{\lambda}, P_{(1,1)} P_{\mu} \rangle} = \frac{\sum_{\sigma=\mu+\square} c^{\lambda}_{\sigma,1} c^{\sigma}_{\mu,1}}{c^{\lambda}_{\mu,(1,1)}} = \frac{c^{\mu+\square_1+\square_2}_{\mu+\square_2,1} c^{\mu+\square_2}_{\mu,1} + c^{\mu+\square_1+\square_2}_{\mu+\square_1,1} c^{\mu+\square_1}_{\mu,1}}{c^{\mu+\square_1+\square_2}_{\mu,(1,1)}} \tag{36}$$

Depicted version of Peri rule



References

[1] L. Lando, A. Zvonkin , Graphs on Surfaces and Their Applications

[2] L. Cassia, R. Lodin and M. Zabzine, JHEP **10** (2020), 126 doi:10.1007/JHEP10(2020)126 [arXiv:2007.10354 [hep-th]].

[3] C. Itzykson and J.-B. Zuber. Matrix integration and combinatorics of modular groups. Communications in Mathematical Physics, 134(1):197 – 207, 1990. doi: cmp/1104201618. URL https://doi.org/.

Jack polynomials

Jack Polynomials is beta deform extension of Schur polynomials. We can define Jack polynomial P similar to Schur polynomial by adding an extra weight.

$$P_\lambda = \sum_T \psi_T(\beta) \prod_{s\in\lambda} z_{T(s)}, \tag{37}$$

where extra weight ψ_T is given with respect to sequence of partition in Young diagram, $\emptyset = \nu_1 \to \nu_2 \to \cdots \to \nu_n = \lambda$.

$$\psi_T(\beta) = \prod_i \psi_{\nu_{i+1}/\nu_i} \qquad \text{where,} \tag{38}$$

$$\psi_{\lambda/\mu} = \prod_{s\in R_{\lambda/\mu} - C_{\lambda/\mu}} \frac{arm_\mu(s) + \beta(leg_\mu(s)+1)}{arm_\mu(s) + \beta leg_\mu + 1} \frac{arm_\lambda(s) + \beta(leg_\lambda(s)+1)}{arm_\lambda(s) + \beta leg_\lambda + 1} \tag{39}$$

where arm(s) is number of boxes in the right of s and leg(s) is number of boxes below s. Jack polynomials form an orthogonal basis $\langle P_\lambda, P_\mu\rangle = 0$ whenever $\lambda \neq \mu$.

If we put $\beta = 1$,we get $\psi = 1$, and we recover definition of Schur.