Entanglement Negativity in $T\bar{T}$-Deformed CFT_2s and Holography

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funded by NARODOWE CENTRUM NAUKI
grant 2021/42/E/ST2/00234

Based on Phy. Rev. D 107 (2023) 12, 126026 with D. Basu and B. Paul

June 08, 2024

Outline

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Introduction and Motivation

Entanglement in Quantum Field Theories

Conformal Field Theories (CFTs)

- Highly constrained from symmetries
- Critical phenomena in phase transitions

$T\bar{T}$- Deformed CFTs

- Irrelevant deformation with exactly solvable spectrum
- Aid from CFT machinery
- Generalization of AdS/CFT Correspondence

Objective: To understand the mixed state entanglement structure of quantum states in $T\bar{T}$ - Deformed CFTs

Outline

Overview of entanglement measures

- Consider a bipartite system $A \cup B$ in a pure state specified by the density matrix ρ.
- Entanglement entropy (EE): von Neumann entropy of the reduced density matrix $\rho_A = \mathrm{Tr}_B(\rho)$,
$$S(A) = -\,\mathrm{Tr}\,\rho_A \ln \rho_A .$$
- EE is a good entanglement measure only for pure quantum states.
- Now, consider a bipartite mixed state ρ_A, where $A = A_1 \cup A_2$.
- EE fails to characterize the entanglement between A_1 and A_2 as it mixes the classical and quantum correlations [Calabrese and Cardy 09'].
- Entanglement negativity (EN): The logarithmic entanglement negativity characterizes the entanglement in the mixed state ρ_A [Vidal and Werner 02'],
$$\mathcal{E} \equiv \ln \mathrm{Tr}\,|\rho_A^{T_2}|. \tag{1}$$

◇ Here the partial transpose of ρ_A with respect to A_2 is given by,
$$\langle e_i^{(1)} e_j^{(2)}|\,\rho_A^{T_2}\,|e_m^{(1)} e_n^{(2)}\rangle = \langle e_i^{(1)} e_n^{(2)}|\,\rho_A\,|e_m^{(1)} e_j^{(2)}\rangle\,.$$

Outline

Part I: Field theoretic calculations of the Entanglement Negativity

Entanglement Negativity in CFT_{1+1}: Replica technique

- The EN is difficult to compute for systems with infinite degrees of freedom such as conformal field theories.
- Using the path integral formulation of density matrices, a *replica technique* was developed to calculate the entanglement measures such as EE and EN [Calabrese and Cardy 09', Calabrese et al. 12'].

Methodology

- Consider the ground state of a CFT with a central charge c, defined on a two-dimensional complex plane $\mathcal{N}$ with coordinates (x, τ).
- We wish to compute the EN for the mixed state ρ_A such that $A_1 = [u_1, v_1]$, $A_2 = [u_2, v_2]$ are two disjoint spatial intervals and $B = A^c$, as depicted in the figure below,

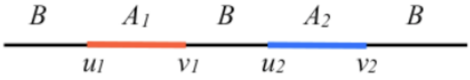

Figure: Two disjoint spatial intervals A_1, A_2 (figure taken from [Calabrese et al. 12']).

- Take n identical copies of the system and $\mathrm{Tr}(\rho_A^{T_2})^n$ is given by,

$$\mathrm{Tr}(\rho_A^{T_2})^n = \frac{Z[\mathcal{N}_n]}{(Z[\mathcal{N}])^n}, \tag{2}$$

where $Z[\mathcal{N}_n]$ and $Z[\mathcal{N}]$ are the partition functions on the replica manifold and the original manifold respectively.

- Finally, the partition function in eq. (2) may be recasted into the correlation function of primary operators on a complex plane such that [Calabrese et al. 12'],

$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle_{\mathbb{C}}\,. \tag{3}$$

◇ Here $\mathcal{T}_n$ and $\bar{\mathcal{T}}_n$ are the twist and anti-twist operators placed on the end points of the intervals A_1 and A_2 with their conformal dimensions given by [Calabrese and Cardy 04', Calabrese and Cardy 09'],

$$h_{\mathcal{T}_n} = h_{\bar{\mathcal{T}}_n} = \frac{c}{12}\left(n - \frac{1}{n}\right). \tag{4}$$

- The EN may be determined as,

$$\mathcal{E} = \lim_{n \to 1} \ln \mathrm{Tr}(\rho_A^{T_2})^n, \quad \forall n \in 2\mathbb{Z}. \tag{5}$$

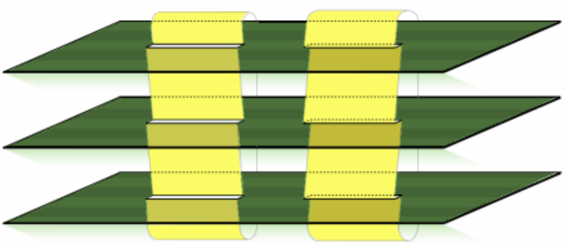

Figure: Path integral representation of $\mathrm{Tr}(\rho_A^{T_2})^n$ for $n = 3$. [Calabrese et al. 12']

$\mathrm{T\overline{T}}$-Deformed $\mathrm{CFT_2}$s

- Describes an *irrelevant* deformation of a 2D CFT by a double trace operator comprised of stress energy tensors.
- The action of a $\mathrm{T\overline{T}}$-Deformed $\mathrm{CFT_2}$ is described by a flow equation with a deformation parameter μ [Zamolodchikov 04']:

$$\frac{\partial S^{(\mu)}_{\mathrm{QFT}}}{\partial\mu} = \int d^2w\,(T_{ww}T_{\bar{w}\bar{w}} - T^2_{w\bar{w}})_\mu\,,\; S^{(\mu)}_{\mathrm{QFT}}\Big|_{\mu=0} = S_{\mathrm{CFT}}.$$

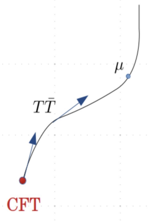

Figure: RG flow of a CFT from a fixed point because of the $\mathrm{T\overline{T}}$ operator.

- Perturbatively, for a small deformation parameter μ

$$S^{(\mu)}_{\mathrm{QFT}} = S_{\mathrm{CFT}} + \mu\int d^2w\,(T_{ww}T_{\bar{w}\bar{w}} - T^2_{w\bar{w}})_{\mu=0}\,.$$

Entanglement Negativity in T$\overline{\text{T}}$-Deformed CFT$_2$s

- Consider a T$\overline{\text{T}}$-deformed CFT$_2$ in an excited state at a finite temperature $1/\beta$ living on a cylindrical manifold $\mathcal{M}$.
- We setup the coordinates on $\mathcal{M}$ as $w = x + i\tau$ and $\bar{w} = x - i\tau$, where $x \in (-\infty, \infty)$ and $\tau \in [0, \beta]$ with $\tau \sim \tau + \beta$.
- Applying the replica technique, the partition function of the deformed theory (on the Riemann surface $\mathcal{M}_n$) in the path integral representation:

$$Z[\mathcal{M}_n] = \int_{\mathcal{M}_n} \mathcal{D}\phi \, e^{-S^{(\mu)}_{\mathrm{QFT}}[\phi]}. \tag{6}$$

- Using eq. (2) and the action of T$\overline{\text{T}}$-deformed CFT, we obtained the EN between two spatial intervals A and B as,

$$\mathcal{E}^{(\mu)}(A:B) = \mathcal{E}_{\mathrm{CFT}}(A:B) + \lim_{n\to 1} \log \left[\frac{\left(1 - \mu \int_{\mathcal{M}_n} \langle T\bar{T} \rangle_{\mathcal{M}_n}\right)}{\left(1 - \mu \int_{\mathcal{M}} \langle T\bar{T} \rangle_{\mathcal{M}}\right)^n} \right]. \tag{7}$$

- Upto the first order of the deformation parameter μ [Lavish et al. 23'],

$$\mathcal{E}^{(\mu)}(A:B) = \mathcal{E}_{\mathrm{CFT}}(A:B) - \mu \lim_{n\to 1} \left[\int_{\mathcal{M}_n} \langle T\bar{T} \rangle_{\mathcal{M}_n} - n \int_{\mathcal{M}} \langle T\bar{T} \rangle_{\mathcal{M}} \right]. \tag{8}$$

Application to various configurations

- To calculate the EN for two disjoint spatial intervals $A = [x_1, x_2]$ and $B = [x_3, x_4]$ in a thermal $\mathrm{T\overline{T}}$-deformed CFT_2, essentially we need to solve for $\left\langle T\bar{T}\right\rangle_{\mathcal{M}_n}$.

$$\int_{\mathcal{M}_n} \langle T\bar{T}\rangle_{\mathcal{M}_n} = \int_{\mathcal{M}} \frac{1}{n} \frac{\left\langle T^{(n)}(w)\bar{T}^{(n)}(\bar{w})\mathcal{T}_n(w_1,\bar{w}_1)\bar{\mathcal{T}}_n(w_2,\bar{w}_2)\bar{\mathcal{T}}_n(w_3,\bar{w}_3)\mathcal{T}_n(w_4,\bar{w}_4)\right\rangle_{\mathcal{M}}}{\left\langle \mathcal{T}_n(w_1,\bar{w}_1)\bar{\mathcal{T}}_n(w_2,\bar{w}_2)\bar{\mathcal{T}}_n(w_3,\bar{w}_3)\mathcal{T}_n(w_4,\bar{w}_4)\right\rangle_{\mathcal{M}}} .$$

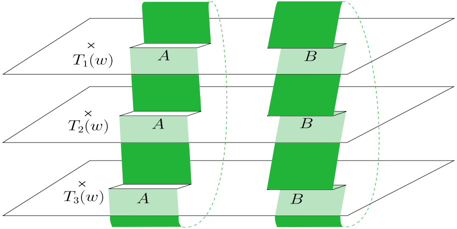

Figure: A schematic of the Riemann surface ($n = 3$) for two disjoint intervals in a $\mathrm{T\overline{T}}$-deformed CFT_2.

- Next we utilize the Ward identities for the stress energy tensor and the large central charge analysis of four point functions to obtain the first order correction in the EN:

$$\delta\mathcal{E}(A:B) = -\frac{\mu c^2\pi^4}{12\beta^3}\left[x_{31}\coth\frac{\pi x_{31}}{\beta} + x_{42}\coth\frac{\pi x_{42}}{\beta} - x_{41}\coth\frac{\pi x_{41}}{\beta} - x_{32}\coth\frac{\pi x_{32}}{\beta}\right] + \delta\mathcal{E}_{\text{cross}}\,.$$

- Utilizing our general formula for the EN, we also determine the EN for the case of two adjacent intervals and a single interval.
- For two adjacent intervals $A = [x_1, x_2]$ and $B = [x_2, x_3]$,

$$\delta\mathcal{E}(A:B) = -\frac{\mu c^2\pi^4}{12\beta^3}\left[x_{21}\coth\frac{\pi x_{21}}{\beta} + x_{32}\coth\frac{\pi x_{32}}{\beta} - x_{31}\coth\frac{\pi x_{31}}{\beta}\right] + \delta\mathcal{E}_{\text{cross}}\,.$$

- For a single interval of length ℓ,

$$\delta\mathcal{E}(A:A^c) = -\frac{\mu\pi^4c^2\ell}{6\beta^3}\left[-1+\coth\frac{\pi\ell}{\beta} - e^{-\frac{2\pi\ell}{\beta}}\frac{f'\left(e^{-\frac{2\pi\ell}{\beta}}\right)}{f\left(e^{-\frac{2\pi\ell}{\beta}}\right)}\right] + \delta\mathcal{E}_{\text{cross}}\,,$$

here f is a function of the cross ratios.

- Notice that the usual UV divergence of the EN is still encoded in the CFT result, the first order correction in the EN due to $T\overline{T}$ deformation is finite.

Outline

Part II: Holographic calculations of the Entanglement Negativity

Holographic Duality

- The holographic duality or AdS/CFT correspondence conjectures that a theory of gravity in a $d+1$ - dimensional Anti de-Sitter spacetime is completely equivalent to a d - dimensional quantum (conformal) field theory living on the boundary of the AdS spacetime.
- A natural implication of the holographic duality is that the observables on one side have a one to one correspondence with the quantities on the other side.

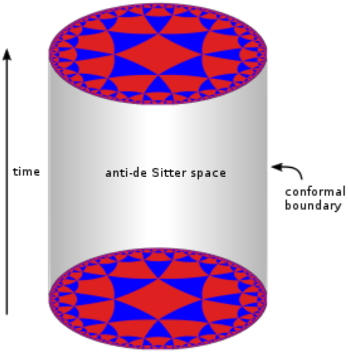

Figure: Holographic spacetime.

Holographic Entanglement Entropy: A brief overview

- The Ryu-Takayanagi proposal states that the entanglement entropy of a subregion A in the boundary CFT is given by the area of a static bulk minimal codimension-2 surface $\tilde{A}$ homologous to A as [Ryu and Takayanagi 06'],

$$S_A = \frac{1}{4G_N}\text{Area}(\tilde{A}).$$

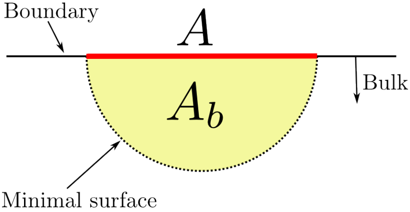

Figure: Holographic prescription for the entanglement entropy (figure taken from [Faulkner et al. 13']).

Cut-off AdS_3/$T\overline{T}$- deformed CFT_2 Proposal and the Holographic Entanglement Entropy

- According to the holographic proposal [Verlinde et al. 18'], a $T\overline{T}$-Deformed CFT_2 (with $\mu > 0$) is dual to an AdS_3 geometry with a finite radial cutoff $r_C = \sqrt{\frac{6R^4}{\pi c \mu}}$.
- The thermal CFT_2 with $T\overline{T}$-deformation is dual to a BTZ black hole (of horizon radius r_H) in the finite radius bulk geometry, with the metric

$$\mathrm{d}s^2 = \frac{r^2 - r_H^2}{R^2}\mathrm{d}t^2 + \frac{R^2}{r^2 - r_H^2}\mathrm{d}r^2 + r^2\mathrm{d}\tilde{x}^2 \,. \tag{9}$$

- The dual $T\overline{T}$-deformed CFT_2 is located at the cut-off radius r_C and hence the metric of the background manifold is conformal to the flat metric as follows

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \frac{\mathrm{d}\tilde{x}^2}{1 - \frac{r_H^2}{r_C^2}} \equiv \mathrm{d}t^2 + \mathrm{d}x^2 \,, \tag{10}$$

where $x = \tilde{x}\left(1 - \frac{r_H^2}{r_C^2}\right)^{-1/2}$ is the spatial coordinate in the CFT_2.

- The authors in [Chen et al. 18', Jeong et al. 19'] showed that the Ryu -Takayanagi formula still holds in the dual finite radius geometry for the holographic EE of bipartite pure states in a $T\overline{T}$-deformed CFT_2 at high temperatures.
- The length of the minimal spacelike surface (geodesic) homologous to a subsystem $A = [x_i, x_j]$ in the deformed CFT_2 at a temperature $1/\beta$ was computed to be [Chen et al. 18', Jeong et al. 19'],

$$\mathcal{L}_{ij} = R \log \left(\mathcal{A}(x_i, x_j) + \sqrt{\mathcal{A}(x_i, x_j)^2 - 1} \right) , \tag{11}$$

where

$$\mathcal{A}(x_i, x_j) \equiv 1 + \frac{2\, r_C^2}{r_H^2} \sinh^2 \left(\frac{\pi |x_i - x_j|}{\beta} \sqrt{1 - \frac{r_H^2}{r_C^2}} \right) . \tag{12}$$

- The holographic EE for various bipartite states may be calculated using the above expression for geodesic lengths.

Holographic Entanglement Negativity in $T\bar{T}$-Deformed CFT_2s

- The holographic construction for the EN of two disjoint intervals A and B in a CFT_2 concerns an algebraic sum of the lengths of bulk minimal spacelike geodesics homologous to various combination of subsystems [Malvimat et al. 18', D. Basu et al. 20'],

$$\mathcal{E}(A:B) = \frac{3}{16G_N}\left(\mathcal{L}_{A\cup C} + \mathcal{L}_{B\cup C} - \mathcal{L}_C - \mathcal{L}_{A\cup B\cup C}\right), \tag{13}$$

where C is another interval sandwiched between A and B.

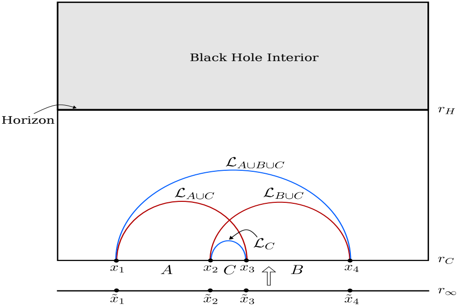

Figure: Ryu-Takayanagi surfaces for two disjoint intervals in a finite radius bulk geometry.

- Utilizing the above construction for the holographic entanglement negativity, we obtain the EN for two disjoint intervals A and B as [Lavish et al. 23'],

$$\mathcal{E}^{(\mu)}(A:B) = \frac{3R}{16G_N} \log \left[\frac{\left(\mathcal{A}(x_1,x_3) + \sqrt{\mathcal{A}(x_1,x_3)^2 - 1}\right)\left(\mathcal{A}(x_2,x_4) + \sqrt{\mathcal{A}(x_2,x_4)^2 - 1}\right)}{\left(\mathcal{A}(x_2,x_3) + \sqrt{\mathcal{A}(x_2,x_3)^2 - 1}\right)\left(\mathcal{A}(x_1,x_4) + \sqrt{\mathcal{A}(x_1,x_4)^2 - 1}\right)} \right] . \tag{14}$$

- Upon solving, it precisely matches with the field theoretical result.
- In a similar manner, we computed the holographic entanglement negativity for the configuration of a single interval and two adjacent intervals and found an agreement from the field theory side [Lavish et al. 23'].

Outline

Conclusions and Future Directions

- We constructed a general formula for the EN of arbitrary bipartite mixed states in $T\overline{T}$-deformed CFT_2, up to first order of the deformation parameter μ.
- Our EN results perfectly matches with certain combinations of geodesic lengths in finite radius AdS_3 BTZ black hole background verifying the Cut-off $AdS_3/T\overline{T}$-deformed CFT_2 proposal of [Verlinde et al. 18'].
- **Future directions:**
  - It will be interesting to apply our work in studying near critical phenomena relevant to phase transitions in corresponding condensed matter systems.
  - Other integrable deformations of CFTs may be explored, possibly leading to the construction of insightful dual gravitational theories and hence, finding more examples of the holographic principle.

Thank You!

Backup

- For a general density operator (ρ) in a basis of quantum states $\{|a\rangle\}$, $\rho = \sum_a p_a |a\rangle\langle a|$, where p_a is the probability corresponding to the state $|a\rangle$. If $\rho^2 = \rho$, $\mathrm{Tr}(\rho^2) = 1 \implies$ system is in a pure state. If $\rho^2 \neq \rho$, $\mathrm{Tr}(\rho^2) < 1 \implies$ system is in a mixed state.
- The positive partial transpose criteria for the separability of mixed states: if a bipartite mixed state is separable, then the partial transpose of its density matrix (ρ^{PT}) is non-negative.
- Path integral representation of density matrices:

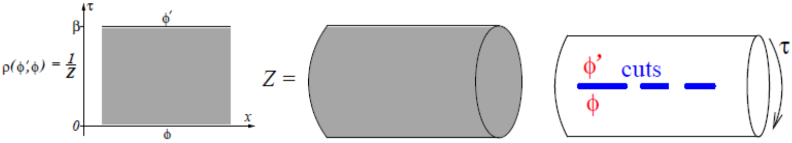

Figure 1: First, representation of a density matrix ρ. Second, the partition function $Z = \mathrm{Tr}(\rho)$. Finally, the reduced density matrix ρ_A where $A = [u_1, v_1] \cup [u_1, v_1] \cup \ldots$ (figure taken from [Calabrese and Cardy 09']).