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# Link Complement States and Entanglement Measures

Abhigyan Saha
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Paweł Caputa, Piotr Sułkowski, Souradeep Purkayastha.

An $n$-component link $\mathcal{L}^n$ is a disjoint union $\mathcal{L}^n = \bigsqcup_{i=1}^{n} \mathcal{K}_i$ of knots $\mathcal{K}_i$, each of which we 'thicken' $^{\mathrm{T}}$ to form a tubular structure diffeomorphic to a solid torus. The manifold $M = S^3 \backslash \mathcal{L}^{n\mathrm{T}}$ thus constructed has a boundary $\partial M = \partial \mathcal{L}^{n\mathrm{T}} = \cup_{i=1}^{n} \partial \mathcal{K}_i^{\mathrm{T}}$.

$$\Psi_{(R_1,L_1),\cdots,(R_n,L_n)}[A^{(0)}] = \int_{A|_{\Sigma}=A^{(0)}} [DA] e^{iS_{CS}[A]} W_{R_1}(L_1) \cdots W_{R_n}(L_n) \tag{1}$$

where $R_{j_i}$ is the $j_i$-th integrable representation of gauge group $G$ corresponding to the value of $k$. [Witten '89]

$$W_{R_{j_i}}(\mathcal{K}_i) = \mathrm{Tr}_{R_{j_i}} \mathcal{P} \exp\left( i \oint_{\mathcal{K}_i} A \right) \tag{2}$$

is a Wilson loop, the trace over understood to be in the $j_i$-th representation and the line integral over $\mathcal{K}_i$ to be path-ordered. The link complement states are then given by the coloured Jones polynomials for $G = \mathrm{SU}(2)$

$$|\mathcal{L}^n\rangle = \sum_{j_1, j_2, \cdots, j_n} C^{\mathcal{L}^n}_{j_1 j_2 \cdots j_n} |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_n\rangle \tag{3}$$

# Entropy measures

We consider some abstract Hilbert space $\mathcal{H}$ that decomposes: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Next, we pick a pure quantum state $|\psi\rangle$ in $\mathcal{H}$ to define the reduced density matrix of $A$ by tracing over the complement $B$: $\rho_A = \mathrm{Tr}_B(\rho)$. Then the von-Neumann entropy:

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \ln \rho_A) = -\sum_i \lambda_i \ln \lambda_i, \tag{4}$$

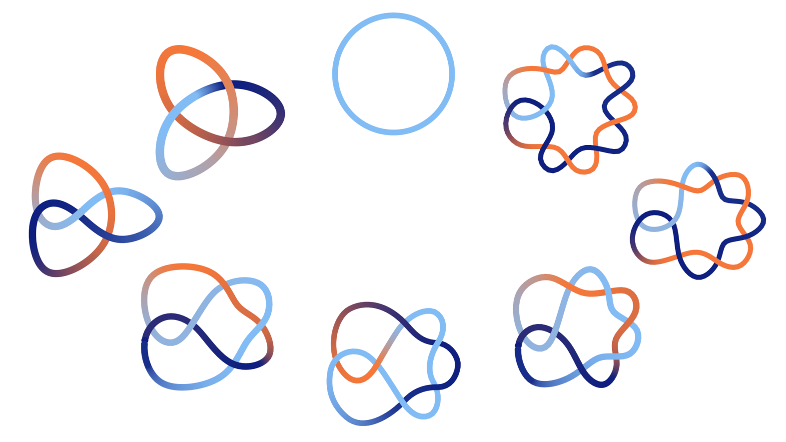
where $\lambda_i$ are the eigenvalues of $\rho_A$. For the quantities computed here, we need two pure states $|\varphi\rangle$ and $|\psi\rangle$ in our $\mathcal{H}$ satisfying $\langle\varphi|\psi\rangle \neq 0$.

Then, we define a transition matrix

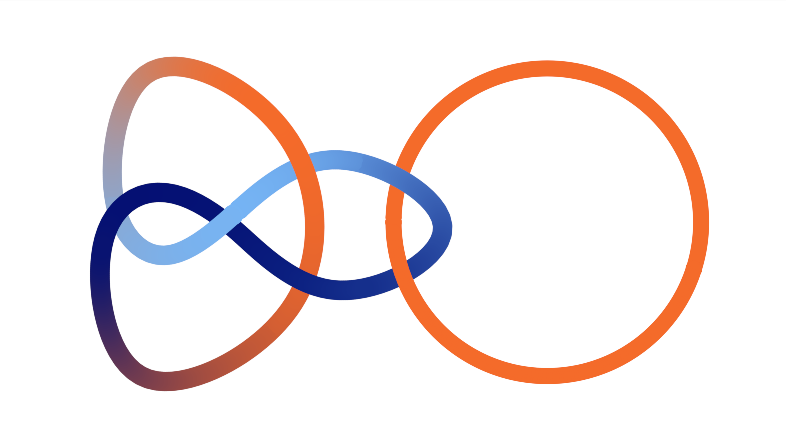
$$\tau^{\varphi|\psi} = \frac{|\varphi\rangle\langle\psi|}{\langle\psi|\varphi\rangle}, \tag{5}$$

and analogous reduced transition matrix for $A$: $\tau_A^{\varphi|\psi} = \mathrm{Tr}_B(\tau^{\varphi|\psi})$.

- **Pseudo Entanglement Entropy** $S_P$**:** complex valued measure obtained by considering the eigenvalues of $\tau_A^{\varphi|\psi}$ in (4). [arXiv:2107.01797]
- **SVD Entanglement Entropy** $S_{SVD}$**:** real valued measure obtained by considering the singular values of $\tau_A^{\varphi|\psi}$ in (4). [arXiv:2307.06531]



Twist knots $\mathcal{K}_p$



Link of type $\mathcal{K}_p\#2^2_1$

These link complement states can be written in the form

$$|\Psi\rangle = \sum_{n=0}^{d} c_n \, |n_1\rangle \otimes |n_2\rangle \tag{6}$$

Torus link $T(P, Q)$

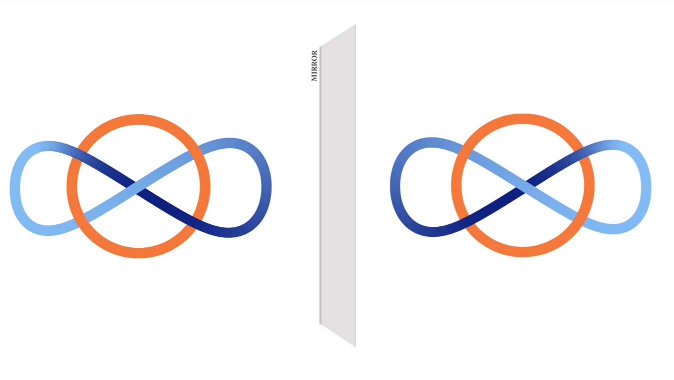
The Pseudo and SVD entanglement entropies for the above infinite families of links have the general closed form expressions:

$$S_{\mathrm{P}}^{1|2} = -\sum_{m=0}^{d-1} \frac{\Theta_m^{1|2}}{\sum_{n=0}^{d-1} \Theta_n^{1|2}} \log\left(\frac{\Theta_m^{1|2}}{\sum_{n=0}^{d-1} \Theta_n^{1|2}}\right), \tag{7}$$

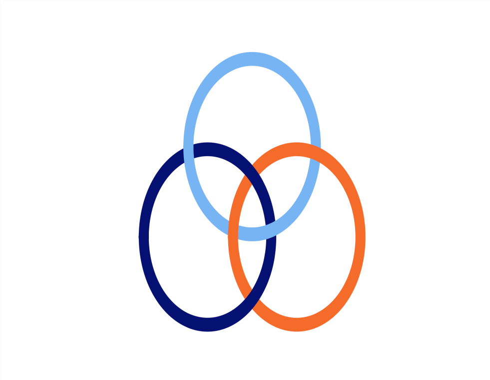
$$S_{\mathrm{SVD}}^{1|2} = -\sum_{m=0}^{d-1} \frac{|\Theta_m^{1|2}|}{\sum_{n=0}^{d-1} |\Theta_n^{1|2}|} \log\left(\frac{|\Theta_m^{1|2}|}{\sum_{n=0}^{d-1} |\Theta_n^{1|2}|}\right), \tag{8}$$

$$\sum_i \langle \psi_i | \overline{a_i} \xleftarrow{\text{Dual}} \sum_i a_i |\psi_i\rangle \xrightarrow{\text{Chiral}} \sum_i \overline{a_i} |\psi_i\rangle$$

A knot $\mathcal{K}$ is chiral if it not is isotopic to its mirror image $\mathcal{K}^\star$; amphichiral otherwise.

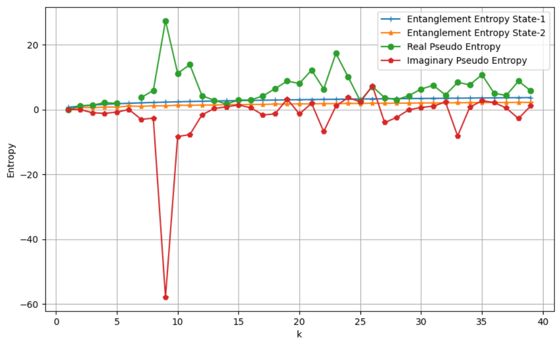



Whitehead link(s)

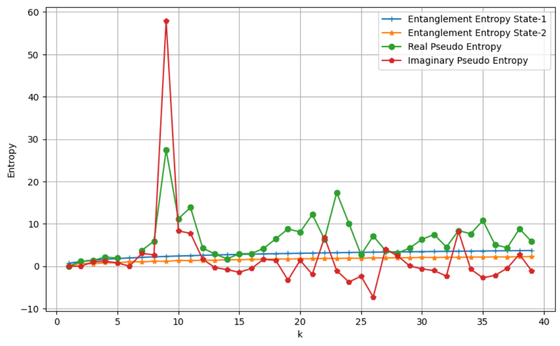


Borromean link

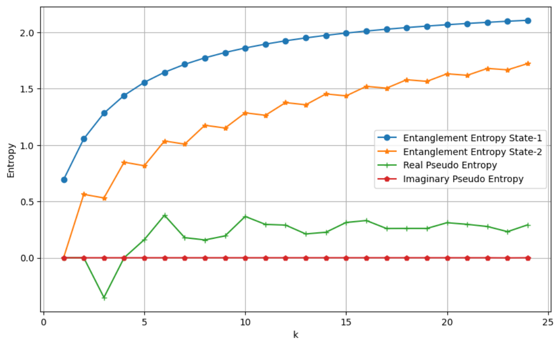
$$|\mathcal{L}\rangle \longrightarrow |\mathcal{L}^{\star}\rangle \implies C^{\mathcal{L}}_{mn}(q) \longrightarrow C^{\mathcal{L}^{\star}}_{mn}(q) = C^{\mathcal{L}}_{mn}(q^{-1}) = C^{\mathcal{L}}_{mn}(\overline{q}) = \overline{C^{\mathcal{L}}_{mn}(q)} \tag{9}$$




$|\mathcal{H}\rangle\langle\mathcal{W}|$




$|\mathcal{H}\rangle\langle\mathcal{W}^{\star}|$




$|\mathcal{H}\#\mathcal{H}\rangle\langle\mathcal{B}|$

# Future directions

Other invariants, such as Unit Invariant Singular Value Decomposition. Pseudo- and SVD entanglement entropies are invariant under similarity and unitary transformations respectively. UISVD is invariant under diagonal transformations. Physically interesting quantity?

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