Exactly solvable models and self-duality in the problem of two linked polymer rings

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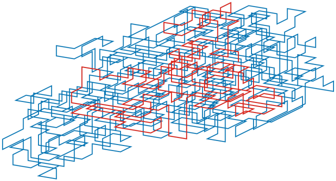

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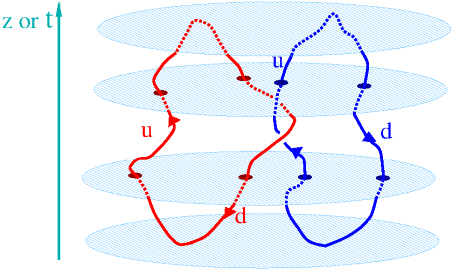
Polymers as quasiparticles

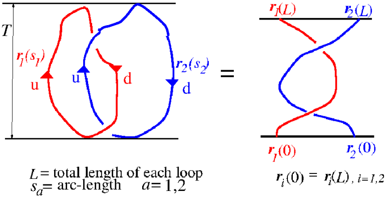


Evolution of $2d-$quasi-particles (vortices) in time t or statistical mechanics in the space $x, y, z = t$ of two linked polymers whose conformations are bound to have one minimum and one maximum each ($4-$plat)

Constraints: The points of maxima and minima can be fixed or not, but there must be only four of such points to have a $4-$plat. Moreover, topological constraints are present.

Polymer-quasiparticle correspondence

The correspondence becomes evident passing from paths...



...to fields:

$$S_{matter} = \sum_{a=1}^{2} \int_0^T dt \int d^2x \left[\vec{\Psi}_a^{*,u} \partial_0 \vec{\Psi}_a^u + |(\boldsymbol{\nabla} - i\gamma_{ab}\boldsymbol{B}^b)\vec{\Psi}_a^u|^2 + \begin{array}{c} u \to d \\ \gamma_{ab} \to -\gamma_{ab} \end{array} \right]$$

$$\gamma_{ab} = \begin{pmatrix} 0 & \lambda \\ \frac{1}{4\pi} & 0 \end{pmatrix}, \qquad \vec{\Psi}_a^*, \vec{\Psi}_a = n\text{−components complex scalar fields}$$

$$B^a = \partial_1 B_2 - \partial_2 B_1 = \gamma^{ba}(|\vec{\Psi}_b^d|^2 - |\vec{\Psi}_b^u|^2)$$

What does it mean self-duality in the case of polymers?

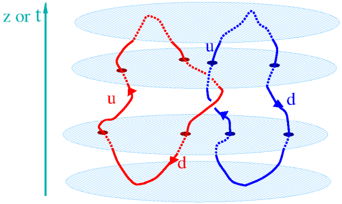
Roughly, $|\vec{\Psi}_a^{u,d}|^2$ is connected with the monomer density of ring a, $a = 1, 2$, *up* or *down* paths.

The magnetic fields $\boldsymbol{B}^a$ intermediate the topological interactions.

λ roughly defines the strength of the topological interactions.

After a Bogomol'nyi transformation the action S_{matter} splits into a self-dual and a non self-dual contributions.

The topology is imposed locally by preventing two polymer lines to cross when they are coming near to each other (non self-dual part of the action S_{matter}), but also the global topological state of the two linked rings must be taken into account (the self-dual terms is taking over this task)



Differences from the Abeliah-Higgs model

Static solutions when the height of the system $T >> 1$

$$S_{\mathsf{matter}} \sim I_{\mathsf{sd}} + I_{\mathsf{C}}$$

$$I_{\mathsf{sd}} = \sum_{a=1}^{2} \int \mathrm{d}^2x \int_0^T \mathrm{d}t \left[\frac{1}{4g_{a,u}} \left| \left(D^u_{a,1} + iD^u_{a,2} \right) \mathbf{\Psi}^u_a \right|^2 + \frac{1}{4g_{a,d}} \left| \left(D^d_{a,1} + iD^d_{a,2} \right) \mathbf{\Psi}^d_a \right|^2 \right]$$

$$I_{\mathsf{C}} = \frac{\lambda}{8\pi} \int \mathrm{d}^2x \int_0^T \mathrm{d}t \left[\left(-\frac{1}{g_{1,u}} |\mathbf{\Psi}^u_1|^2 + \frac{1}{g_{1,d}} \left|\mathbf{\Psi}^d_1\right|^2 \right) \left(-|\mathbf{\Psi}^u_2|^2 + \left|\mathbf{\Psi}^d_2\right|^2 \right) + \left(\frac{1}{g_{2,u}} |\mathbf{\Psi}^u_2|^2 - \frac{1}{g_{2,d}} \left|\mathbf{\Psi}^d_2\right|^2 \right) \left(-|\mathbf{\Psi}^u_1|^2 + \left|\mathbf{\Psi}^d_1\right|^2 \right) \right]$$

The non self-dual contributions I_C describe short-range Coulomb-line interactions. When the parameters $g_{a,u}$ and $g_{a,d}$ which take into account the flexibility of the polymers are all equal (homopolymer case) only the self-dual contribution S_{sd} survives.

NOTE: not exactly an Abelian-Higgs model with well specified points of energy minimum (e. g. $|\vec{(\Psi)}_a^{u,d}|^2 = V^2$).

Self-dual solutions minimizing the static energy when $T >> 1$

Replica symmetry breaking solutions, only the first replica sector $\psi_a^{*u,d}, \psi_a^{u,d}$ is taken into account.
Putting (winding number or vorticity $N = 1$)

$$\vec{\psi}_a^{u,d} = e^{i\theta_a^{u,d}} \rho_a^{u,d}$$

self-dual equations provide these relations about the *up* and *down* components:

$$\theta_a^u = -\theta_a^d \quad \text{and} \quad \rho_a^u = \frac{A_a}{\rho_a^d}$$

the $A_a' s$ being constants.
We search for solutions such that $\rho_1^u = \rho_2^u$ (justified by the fact that ρ_1^u and ρ_2^u enter symmetrically in the self-dual equations.

Explicit solutions

The self-dual equation for ρ_1^1 can be reduced to a sinh-Gordon (if $A_1 > 0$) or a cosh-Gordon equation (if $A_1 < 0$).
We search for translationally invariant solutions along x_2. This case becomes realistic in a confined geometry of a box of sides $L_1 \times L_2 \times T$ with $L_2 >> L_1$ and $T >> 1$.
After a rescaling of ρ_1^2 and the introduction of the new parameter m^2:

$$\rho_1^u = \frac{\pi}{\lambda} v_1 \qquad\qquad m^2 = \frac{2\lambda\sqrt{|A_1|}}{\pi}$$

the following equation is obtained:

$$\left(\frac{dv_1}{dx_1}\right)^2 = 2v_1^3 + 2Ev_1^2 - t\frac{m^4}{2}v_1$$

where E =integration constant and $t = \mathrm{sgn}(A_1)$. The above equation can be reduced in the standard Weierstrass form using the substitution:

$$v_1 = 2y - \frac{E}{3}$$

Explicit translationally invariant solutions along x_2

ranges of the y (Weierstrass) function

unbounded solutions	bounded solution
$y \in [e_1, +\infty)$ for $\Delta > 0$ $y \in [e_2, +\infty)$ for $\Delta < 0$	$y \in [e_3, e_2]$ for $\Delta > 0$

(only the real solutions)

The ranges of ρ_1^u and ρ_2^u.

parameters	$\rho_1^u = \rho_2^u = \frac{\pi}{\lambda} v_1$	unbounded range	bounded range
$t = +1$	$\frac{2\pi}{\lambda}(y - e_2)$	$\left[\frac{2\pi}{\lambda}(e_1 - e_2), +\infty\right)$	$\left[-\frac{2\pi}{\lambda}(e_2 - e_3), 0\right]$
$t = -1,\ E > m^2$	$\frac{2\pi}{\lambda}(y - e_1)$	$[0, +\infty)$	$\left[-\frac{2\pi}{\lambda}(e_1 - e_3), -\frac{2\pi}{\lambda}(e_1 - e_2)\right]$
$t = -1,\ E < -m^2$	$\frac{2\pi}{\lambda}(y - e_3)$	$\left[\frac{2\pi}{\lambda}(e_1 - e_3), +\infty\right)$	$\left[0, \frac{2\pi}{\lambda}(e_2 - e_3)\right]$
$t = -1,\ \lvert E \rvert < m^2$	$\frac{2\pi}{\lambda}(y - e_2)$	$[0, +\infty)$	—