Scattering amplitudes from null-cone geometry

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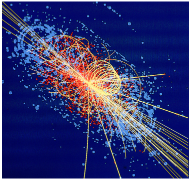
[2306.07312] with J. Stalknecht

[2308.02438] with L. Ferro, R. Glew and J. Stalknecht

Scattering amplitudes in Quantum Field Theories



Recent approaches: geometrization of scattering amplitudes:

Amplitude = Volume of a polytope [Hodges]

= Logarithmic differential form on some “positive” space [Arkani-Hamed, et.al]

Introduction

Playground: Maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ super Yang-Mills (SYM))

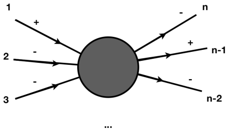
$\longrightarrow$ interacting 4d QFT with highest degree of symmetry

$\longrightarrow$ supersymmetric cousin of Quantum Chromodynamics (QCD)

$\longrightarrow$ simplicity: "Hydrogen atom of the 21st century"

Geometrization of amplitudes in planar $\mathcal{N} = 4$ SYM:

- **Amplituhedron:** $\mathcal{A}^{(m)}_{n,k'}$ [Arkani-Hamed, Trnka]
- **Momentum Amplituhedron:** $\mathcal{M}^{(4)}_{n,k}$ [Damgaard, Ferro, TL, Parisi], [Ferro, TL]



- number of particles: n
- space-time dimension: m – in physics $m = 4$
- type of scattered particles: k' – $(++\underbrace{+\ldots+}_{k'}\underbrace{-\ldots-}_{n-k'-4}--)$

Generalizations of this idea to other models: **positive geometries** [Arkani-Hamed,Bai,Lam]

Standard approach to perturbative scattering amplitudes

- Write down a Lagrangian consistent with the particle content and all symmetries
- Derive Feynman rules for on-shell processes (**LSZ reduction**)
- Evaluate Feynman diagrams contributing to n-particle scattering at a given loop order

$$A_n = A_n^{(0)} + \lambda A_n^{(1)} + \lambda^2 A_n^{(2)} + \ldots$$

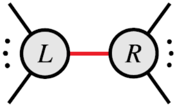
- The final answer is usually much simpler than the intermediate steps because of:
  - **gauge redundancies**
  - **off-shell processes**

Can we find a way to arrive to the final answer in a **simpler, more direct way**?

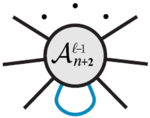
Yes, use properties of amplitudes

- **Locality and unitarity:** scattering amplitudes factorize in smaller pieces on physical poles

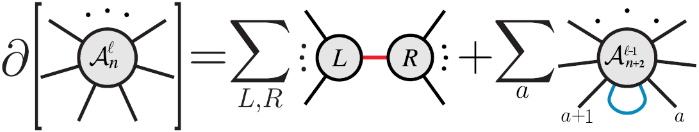
Tree level:



Forward-limits:

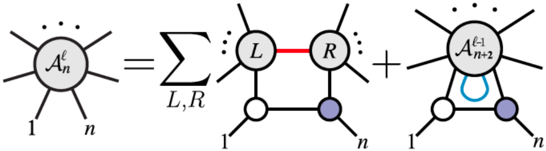


- The **singularities** of full amplitude are

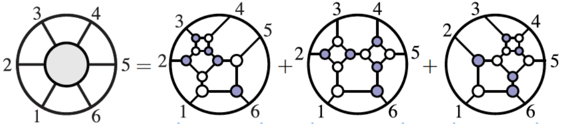


Amplitude: BCFW recursion relations

- Solution for the all-loop planar $\mathcal{N} = 4$ SYM



- Any tree-level amplitude/loop integrand in terms of on-shell diagrams



All propagators are **on-shell** – no off-shell integrations.

- On-shell diagrams composed of two types of three-point amplitudes

Scattering amplitudes as differential forms

- **Price to pay**: recursion relations introduce unphysical poles in each term of the answer (**spurious singularities**). They cancel out in the final answer.

How to encode the structure of singularities of amplitudes in an efficient way?

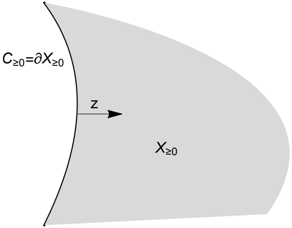
- Tree amplitudes/loop integrands are **rational functions** of kinematic data
- Enhance them to **differential forms**
- The **boundary operator** ∂ corresponds to the operation of taking residues:

$$\operatorname*{Res}_{z\to 0}\frac{dz}{z}\wedge\omega=\omega$$

- We can continue this process to get further factorizations

Appropriate **region in the kinematic space** and its **logarithmic form**
$\rightarrow$ tree-level scattering amplitudes and integrands for loop amplitudes in $\mathcal{N}=4$ SYM

Positive geometries and logarithmic differential forms



Positive geometries: a positive space $X_{\geq 0}$ with a meromorphic form $\Omega(X_{\geq 0})$

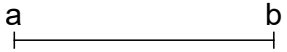
- For $D = 0$: $X_{\geq 0}$ is a single point and $\Omega(X_{\geq 0}) = \pm 1$.
- For $D > 0$: we must have
  - Every **boundary component** $C_{\geq 0}$ of $X_{\geq 0}$ is a positive geometry of dimension $D - 1$
  - The **differential form** on boundaries is given by residue

$$\operatorname*{Res}_{z \to 0} \Omega(X_{\geq 0}) = \mathrm{Res}_{z=0} \left(\frac{dz}{z} \wedge \Omega(C_{\geq 0}) \right) = \Omega(C_{\geq 0})$$

Examples of positive geometries

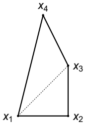
- **Line segment:** $[a,b] \subset \mathbb{R}$

$$\Omega([a,b]) = d\log\frac{z-a}{z-b} = \frac{dz}{z-a} - \frac{dz}{z-b}$$



- **A quadrilateral:** $Q \subset \mathbb{R}^2$:

$$\Omega(Q) = [123] + [134]$$



where

$$[ijk] = d\log\frac{\langle ij\rangle}{\langle ik\rangle} \wedge d\log\frac{\langle jk\rangle}{\langle ik\rangle}$$

and

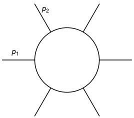
$$\langle ij\rangle = x_i^1 x_j^2 - x_i^2 x_j^1$$

All convex polytopes are positive geometries

Positive geometries generalise convex polytopes to curvy geometric regions

Planar sector of $\mathcal{N} = 4$ SYM

Amplitudes

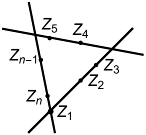


On-shell superspace

$$\boxed{\mathcal{W}_i^{\mathcal{A}} = (\lambda_i^{\alpha}, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_i^{\alpha}, \tilde{\eta}_i^{\dot{\alpha}})}$$

Momentum amplituhedron

Twistor theory

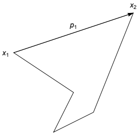


Momentum-twistor superspace

$$\boxed{\mathcal{Z}_i^{\mathcal{A}} = (\lambda_i^{\alpha}, \tilde{\mu}_i^{\dot{\alpha}}, \chi_i^{A}) = (z_i^{A}, \chi_i^{A})}$$

Amplituhedron

Wilson loops



Dual superspace

$$\boxed{\mathcal{X}_i^{\mathcal{A}} = (\lambda_i^{\alpha}, x_i^{\alpha\dot{\alpha}}, \theta_i^{\alpha A})}$$

Null-cone geometries

Dual space

Dual momentum space: **split-signature space** with $(2,2)$ signature $(+,+,-,-)$

4D on-shell momenta p_i^μ subject to p_i^μ and momentum conservation $\sum_i p_i^\mu = 0$

$$\downarrow$$

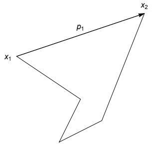
equivalently encoded using **dual momentum coordinates** x^μ defined as

$$p_i^\mu = x_{i+1}^\mu - x_i^\mu$$

Two points x^μ and y^μ are **null-separated** if

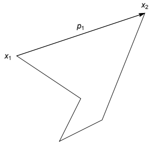
$$(x-y)^2 := (x^1-y^1)^2 + (x^2-y^2)^2 - (x^3-y^3)^2 - (x^4-y^4)^2 = 0$$

The collection of dual momenta x_i^μ defines a null polygon in $\mathbb{R}^{2,2}$:



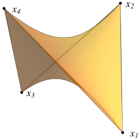
Tree-level

In **dual momentum space** the positive geometry for tree-level amplitudes is a set of null polygons satisfying particular **positivity conditions**.



Loops

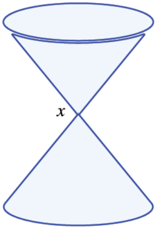
The positive geometry for integrands of loop amplitudes is given by a particular **compact region** containing points that are **positively separated from all vertices** of the null polygon.



3D intuition – null-cones

Null-cone:

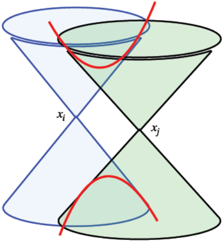
$$\mathcal{N}_x = \{y \in \mathbb{R}^{1,2} : (y-x)^2 = 0\}$$



3D intuition – intersection of two nullcones

Intersection of two null-cones for generic **positively-separated points**:

$$\mathcal{N}_{x_i} \cap \mathcal{N}_{x_j} = \{y \in \mathbb{R}^{1,2} : (y - x_i)^2 = (y - x_j)^2 = 0\}$$



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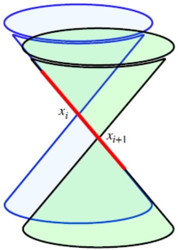
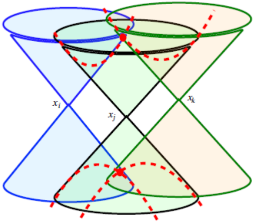


3D intuition – intersection of three nullcones

Intersection of three null-cones for generic **positively-separated points**:

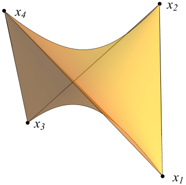
$$\mathcal{N}_{x_i} \cap \mathcal{N}_{x_j} \cap \mathcal{N}_{x_k} = \{y \in \mathbb{R}^{1,2} : (y - x_i)^2 = (y - x_j)^2 = (y - x_k)^2 = 0\} = \{p^+_{ijk}, p^-_{ijk}\}$$



3D intuition – positive geometry for 4 points

Positive geometry for one-loop integrands for 4 points in ABJM theory

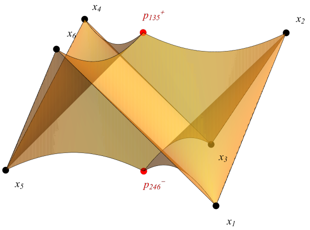
$$\{y \in \mathbb{R}^{1,2} : (y - x_i)^2 > 0, i = 1, 2, 3, 4 + \text{a sign-flip condition}\}$$



3D intuition – positive geometry for 6 points

Positive geometry for one-loop integrands for 6 points in ABJM theory

$$\{y \in \mathbb{R}^{1,2} : (y - x_i)^2 > 0, i = 1, 2, \ldots, 6 + \text{a sign-flip condition}\}$$



Back to $\mathcal{N} = 4$ SYM – one loop

We describe the one-loop geometry using the **null structure** of $\mathbb{R}^{2,2}$

- Define a **non-negative** and a **non-positive** part:

$$\mathcal{N}_x^+ = \{y \in \mathbb{R}^{2,2} : (y-x)^2 \geq 0\}, \qquad \mathcal{N}_x^- = \{y \in \mathbb{R}^{2,2} : (y-x)^2 \leq 0\}$$

- For a **fixed null polygon** with vertices x_i, we define

$$\mathcal{K}_{n,k}(x) := \{y \in \mathbb{R}^{2,2} : (y-x_i)^2 \geq 0 \text{ for all } i = 1, 2, \ldots, n\} = \bigcap_{i=1}^{n} \mathcal{N}_{x_i}^+$$

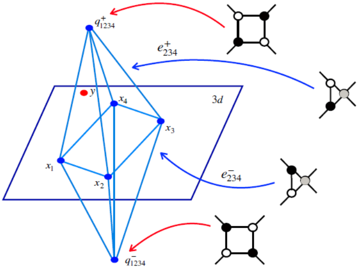
- The region $\mathcal{K}_{n,k}(x)$ decomposes into a **compact** and a **non-compact** regions:

$$\mathcal{K}_{n,k}(x) = \Delta_{n,k}(x) \cup \overline{\Delta}_{n,k}(x)$$

- Points in the compact region $\Delta_{n,k}(x)$ have **correct number of sign flips**!

Example: MHV$_4$

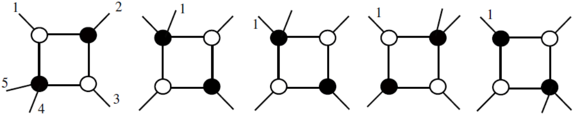
- Vertices of $\Delta_{4,2}$: maximal cuts
- Edges of $\Delta_{4,2}$: triple cuts
- Dim-2 boundaries of $\Delta_{4,2}$: double cuts
- Facets of $\Delta_{4,2}$: single cuts = forward limits



Example: all MHV$_n$

For **higher number of particles** $\rightarrow$ not all quadruple cuts are inside the geometry, e.g. for $n=5$

$$\mathcal{V}(\Delta_{5,2}(x)) = \{q^+_{1234}, q^+_{2345}, q^+_{3451}, q^+_{4512}, q^+_{5123}\}$$



General MHV$_n$:

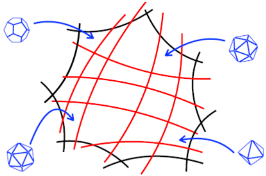
- One-loop geometries **combinatorially identical** for all tree-level configuration of points
- Vertices of one-loop geometry correspond to all **allowed quadruple cuts** of amplitudes

$$\mathcal{V}(\Delta_{n,2}(x)) = \{q^+_{ii+1jj+1}\}_{i,j=1,2,\ldots,n}$$

Beyond MHV

- For $k > 2$ the one-loop geometry **does** depend on the details of the tree-level configuration
- **However**, there are finitely many combinatorially inequivalent geometries

  $\rightarrow$ can be systematically classified
- The complete one-loop geometry (tree-level + one-loop) can be understood as a **fibration**:



Each region where the one-loop geometry is constant is called a **chamber**.

Chambers are the **maximal intersections of BCFW tiles**.

From geometry to integrands

- For a **fixed null polygon** with vertices x_i, one finds a **compact region** $\Delta_{n,k}(x) \in \mathbb{R}^{2,2}$
- Vertices of $\Delta_{n,k}(x)$:
  - the vertices x_i of the null polygon
  - a subset of quadruple intersection of nullcones $q^{\pm}_{ijkl}$
- Every vertex of $\Delta_{n,k}(x)$ has exactly **four edges incident** to it

The one-loop geometry $\Delta_{n,k}(x)$ is a **curvy version of a simple polytope**

- For **simple polytopes** (d-dimensional polytopes with exactly d edges/facets meeting at each vertex), canonical differential form can be written as the **sum over all vertices** of known forms associated to each vertex

Canonical differential forms

- For curvy version of simple polytopes, **a conjecture**:

$$\Omega[\Delta_{n,k}(x)] = \sum_{q^{\pm}_{ijkl} \in \Delta_{n,k}(x)} \omega_{ijkl} + \sum_i \omega_{x_i}$$

where $\omega_{x_i} = 0$ and

$$\omega_{ijkl} = d\log(y - x_i)^2 \wedge d\log(y - x_j)^2 \wedge d\log(y - x_k)^2 \wedge d\log(y - x_l)^2$$

- Almost correct but has **non-zero residues outside the geometry** $\to$ can be corrected:

$$\Omega[\Delta_{n,k}(x)] = \sum_{q^{\pm}_{ijkl} \in \Delta_{n,k}(x)} \omega^{\pm}_{ijkl}$$

where

$$\omega^{\pm}_{ijkl} = \frac{1}{2}(\omega^{\square}_{ijkl} \pm \omega_{ijkl})$$

$$\omega^{\square}_{ijkl} = d\log\frac{(y - x_i)^2}{(y - q^{\pm}_{ijkl})^2} \wedge d\log\frac{(y - x_j)^2}{(y - q^{\pm}_{ijkl})^2} \wedge d\log\frac{(y - x_k)^2}{(y - q^{\pm}_{ijkl})^2} \wedge d\log\frac{(y - x_l)^2}{(y - q^{\pm}_{ijkl})^2}$$

Examples: MHV amplitudes

- For MHV amplitudes ($k = 2$), the one-loop geometry $\Delta_{n,k}(x)$ is combinatorially identical for all tree-level configurations – **only one chamber**
- The integrand of MHV amplitudes:

$$\Omega_{n,2,1} = \Omega_{n,2,0} \wedge \Omega[\Delta_{n,2}]$$

- For example: $\underline{n = 4}$

$$\Omega_{4,2,1} = \Omega_{4,2,0} \wedge \omega^{\square}_{1234}$$

with

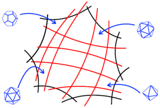
$$\omega^{\square}_{1234} = \frac{d^4y\, x_{13}^2 x_{24}^2}{(y-x_1)^2(y-x_2)^2(y-x_3)^2(y-x_4)^2}$$

which is the well-known box integrand.

- **General MHV:**

$$\Omega[\Delta_{n,2}] = \frac{1}{2}\sum_{i<j} \omega^{\square}_{ii+1jj+1} + \frac{1}{2}\sum_{1<i<j} \omega^{\pentagon}_{1ii+1jj+1}$$

General one-loop answer



1. Classify all chambers $\mathcal{C}_{n,k}$
2. For a chamber $\mathfrak{c}$, take a point $x_{\mathfrak{c}}$ (null polygon) inside and find all vertices in $\Delta_{n,k}(x_{\mathfrak{c}})$
3. The tree-level + one-loop canonical form is

$$\Omega_{n,k,1} = \sum_{\mathfrak{c}\in\mathcal{C}_{n,k}} \Omega^{(0)}_{\mathfrak{c}} \wedge \Omega[\Delta_{n,k}(x_{\mathfrak{c}})]$$

4. Can be rewritten in a simpler way

$$\Omega_{n,k,1} = \sum_{\sigma\in\mathcal{S}_{n,k}} \Omega^{(0)}_{\sigma} \wedge \Omega^{(1)}_{\sigma} = \sum_{\sigma\in\mathcal{S}_{n,k}} \Omega^{(0)}_{\sigma} \wedge \sum_{q^{\pm}_{ijkl}} \omega^{\pm}_{ijkl}$$

All integrands for one-loop amplitudes in planar $\mathcal{N}=4$ SYM!

This formula is **combinatorial** – no need to know details of the geometry

Beyond one loop

- For more loops, the ℓ-loop geometry is a collection of points

$$(y_1, y_2, \ldots, y_\ell)$$

each of them inside the one loop geometry, that are positively separated from each other

$$(y_i - y_j)^2 > 0$$

- One needs to find a proper fibration of the complete ℓ-loop geometry and find a similar combinatorial description of contributions (work in progress)

Summary

- We have translated the momentum amplituhedron, that is the positive geometry for planar $\mathcal{N} = 4$ SYM in spinor helicity space, into the dual momentum space
- Loop geometry depends on the tree configuration
- Within a chamber, the one-loop geometry is combinatorially constant
- One-loop integrand can be found from the canonical differential form of this space
- One-loop geoometry for each null polygon is a curvy version of a simple polytope

Main result

We found a new formula for one-loop integrands for planar $\mathcal{N} = 4$ SYM scattering amplitudes

Outlook

- **For phenomenological applications:** find a way to integrate this integrands in a way compatible with geometry to extract observables
- **For theoretical physicists:** find all integrands at higher loops, for any multiplicity and helicity sector
- **For mathematicians:** many interesting combinatorial questions, eg. boundary stratification of the loop spaces, generalisations of positive Grassmannians, etc.

Thank you for your attention!