Holographic Approach to Scale vs. Conformal Invariance

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Motivation: Conformal Group (and its Subgroups)

Algebra:

$$[P_m, M_{ns}] = 2\eta_{m[n}P_{s]}$$

$$[M_{mn}, M_{rs}] = \eta_{mr}M_{sn} - \eta_{nr}M_{sm} - \eta_{ms}M_{rn} + \eta_{ns}M_{rm}$$

Poincaré Inv.

$$[D, P_m] = P_m$$

Scale without Conformal Inv.

SwCI

$$[K_m, M_{ns}] = 2\eta_{m[n}K_{s]}$$

$$[P_m, K_n] = 2(\eta_{mn}D + M_{mn})$$

$$[D, K_m] = -K_m$$

Other Brackets $= 0$

Conformal

Q: Are there SwCI QFTs ?

In $d=2$, scale implies conformal Inv.

In $d>2$: ?

See: Yu Nakayama, esp. 1302.0884

Q: Can we build SwCI models in Gauge/Gravity Duality ?

Possible Outcomes:

QFT allows SwCI	Bulk allows SwCI	
Yes	Yes	→ Holographic Model Building
No	No	→ Holographic No-Go-Theorems
Yes	No	→ Holographic Theories are special
No	Yes	→ Forbidden Landscape

Either outcome is interesting!

Q: Can we construct a "Bulk" spacetime with Isometry group

(~ Algebra of Killing vectors) ~ SwCl ?

Killing eq: $0 \overset{!}{=} \mathcal{L}_K g_{\mu\nu} = \nabla_\mu K_\nu + \nabla_\nu K_\mu$

Lie-bracket: $[K_1, K_2]^\mu = K_1^\nu \partial_\nu K_2^\mu - K_2^\nu \partial_\nu K_1^\mu = K_3^\mu$

Naive Ansatz: $$ds^2 = e^{2A(w)} \underbrace{(-dt^2 + dx^2)}_{\text{Minkowski}} + dz^2$$

Imposing Scale Inv. fixes $A(w)$

$\Downarrow$

AdS Space

$\Downarrow$

Conf. Inv.

New Ansatz: (Warped) Extra dimensions in the Bulk!

Balasubramanian: 1301.6653 ; Flory: 1711.03113

Ansatz: (can describe SWCl or DSl)

$$ds^2 = e^{2C(w,\theta)}\Big[\underbrace{e^{\frac{2w}{L}}\overbrace{(-dt^2 + d\vec{x}^2)}^{\text{Minkowski}} + dw^2}_{\text{AdS}}\Big] + e^{2B(w,\theta)}\left(d\theta + A_{(w,\theta)}dw\right)^2$$

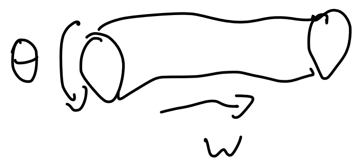
$\theta \in [0, 2\pi[$

$y^i \in \{w, \theta\}$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = S(w,\theta)\overbrace{(-dt^2 + d\vec{x}^2)}^{\text{Minkowski}} + h_{ij}dy^i dy^j$$

"Fiber" "Base"

Topology of cylinder



Killing Vectors: $P_m{}^\mu$, $M_{nr}{}^\mu$ automatic

$$D^\mu = \begin{pmatrix} -t \\ -x \\ \gamma^w \\ \gamma^\theta \end{pmatrix}$$ because of Algebra alone.

Killing eq: $\nabla_\mu D_\nu + \nabla_\nu D_\mu = 0$ 4d Problem for D^μ, $g_{\mu\nu}$

$$\Downarrow$$

$$\left.\begin{aligned} &\mathcal{L}_\gamma h_{ij} = 0 \quad \text{(Killing eq. in 2d Base)} \\ &\mathcal{L}_\gamma S = 2S \end{aligned}\right\}$$ 2d Problem for γ^i, h_{ij}, S

solution		solution
S	$\rightarrow$	$f_1(G) \times S$
h_{ij}	$\rightarrow$	$f_2(G) \times h_{ij}$

for any f_1, f_2, as long as $\mathcal{L}_\gamma G = 0$

But: We want Scale without Conf. Invariance!

$\rightarrow$ No Killing Vector $\sim K_m{}^{\mu}$ should exist

Check Weyl tensor: $\mathcal{L}_K C_{\alpha\beta\gamma\delta} = 0$

Necessary, but not sufficient cond. for Killing Vector K

$C_{\alpha\beta\gamma\delta} = 0$

$\mathcal{L}_K C_{\alpha\beta\gamma\delta} = 0$ trivially

Killing Vector K_m can exist.

$C_{\alpha\beta\gamma\delta} \neq 0$

$C_{\alpha\beta\gamma\delta} \overset{\text{w.p.}}{=} \phi \cdot \tilde{C}_{\alpha\beta\gamma\delta}$

Simple, no derivatives

We can show:

$\underline{\mathcal{L}_K C_{\alpha\beta\gamma\delta} \neq 0}$!

$$\boxed{\text{Weyl} \neq 0 \Rightarrow \text{SwCI !}}$$

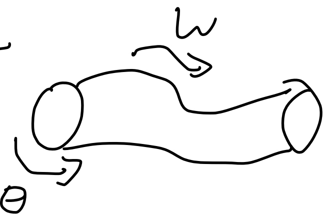
Model Building:

Model: $\{ S, h_{ij}, \xi^i \}$ such that D is killing

Conditions:

1. Weyl $\neq 0 \Rightarrow$ SWCL — Local Geometric Cond.
2. NEC: $T_{\mu\nu} n^\mu n^\nu \geq 0$ — Physical Cond.
3. Extra Dim. compact! — Topological Cond.

$\theta \sim \theta + 2\pi$



Conjecture: (1) + (2) + (3) mutually exclusive!

Partial Proof

Work In Progress!

$\underline{NEC}$:

$$T_{\mu\nu} n^\mu n^\nu \geq 0 \quad \overset{\text{Einstein eq.}}{\Longrightarrow} \quad R_{\mu\nu} n^\mu n^\nu \geq 0$$

$$g_{\mu\nu} \overset{\text{w.p.}}{=} \begin{pmatrix} \overbrace{S\,\eta_{ab}}^{2\times2\text{ Fiber}} & 0 \\ 0 & \underbrace{h_{ij}}_{2\times2\text{ Base}} \end{pmatrix} \Rightarrow R_{\mu\nu} \overset{\text{w.p.}}{=} \begin{pmatrix} X\cdot\eta_{ab} & 0 \\ 0 & r_{ij} \end{pmatrix}$$

$$n^\mu = \begin{pmatrix} \hat{n}^a \\ \bar{n}^i \end{pmatrix} \begin{matrix} \}\ \text{Fiber} \\ \}\ \text{Base} \end{matrix} \qquad n^\mu n_\mu = 0 \Rightarrow \hat{n}^a \hat{n}^b \eta_{ab} = -\frac{\bar{n}^i \bar{n}^j h_{ij}}{S}$$

$$\Rightarrow NEC: \quad 0 \leq R_{\mu\nu} n^\mu n^\nu = \bar{n}^i \bar{n}^j \Big(\underbrace{r_{ij} - \frac{X}{S} h_{ij}}_{\equiv t_{ij}} \Big)$$

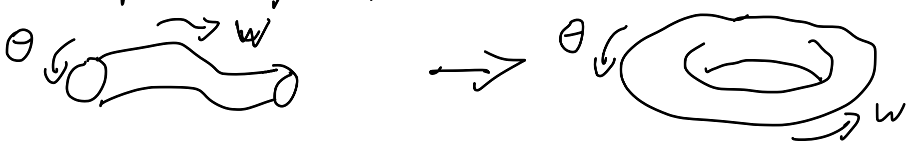
$$\boxed{\text{NEC} \iff t^i{}_j \text{ is a positive semi-def. } 2\times 2 \text{ matrix}}$$

$$\mathrm{Tr}\left[t^i{}_j\right] = \bar{R} + \frac{2}{s}\left(\bar{\nabla}\sqrt{s}\right)^2 \qquad \bar{R} = R[h] \text{ in 2d!}$$

$$\mathrm{Det}\left[t^i{}_j\right] = \frac{1}{4}\left[\bar{R} + \frac{2}{s}\left(\bar{\nabla}\sqrt{s}\right)^2\right] + \frac{1}{s}\left(\bar{\Box}\sqrt{s}\right)^2 - \frac{2}{s}\left(\bar{\nabla}_i\bar{\nabla}_j\sqrt{s}\right)\left(\bar{\nabla}^i\bar{\nabla}^j\sqrt{s}\right)$$

<u>Naive Idea:</u> Just make $\bar{R} > 0$ everywhere (and $\bar{R}$ big enough) to guarantee NEC ($\Rightarrow \det > 0$, $\mathrm{tr} > 0$)

<u>But!</u> We can compactify cylinder to torus b.c. of Killing Vector J:



<u>Gauss-Bonnet:</u> $\int \bar{R}\, dV \sim 2\pi\chi = \underline{\underline{0}} \Rightarrow \bar{R} > 0$ everywhere is not possible!

Outlook / To Dos:

- Finalise Proof
- What if extra dim is not compact?
  E.g. Defect / Janus like models?

Paper Soon

- How much changes as dimension of base and fiber changes?

Thank you very much!