Generalized hydrodynamics of Integrable QFTs

Jacek Pawełczyk

University of Warsaw, Poland

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Plan

- Relativistic integrable QFT theories
- Thermodynamics of integrable theories
- Thermodynamics of integrable QFT in finite volume
- Hydrodynamics of integrable theories
- Hydrodynamics of integrable QFT in finite volume

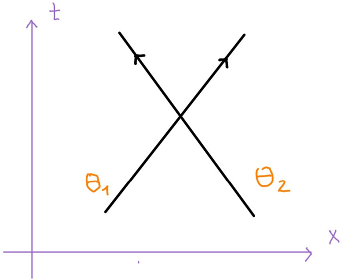
based on arXiv:2308.05010+new stuff

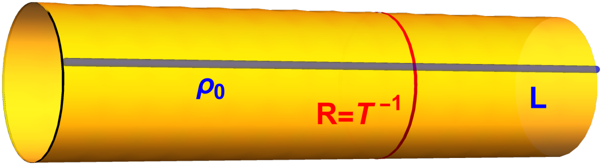
Thanks to Milosz Panifil.

$E^2 + p^2 = m^2$
$E(\theta) = m\cosh(\theta),\ p(\theta) = \sinh(\theta)$
integrable: scattering phase

$\Phi(\theta_1 - \theta_2) = -i\log(S_{scatt})$

$\varphi(\theta) = \partial_\theta \Phi = \Phi'(\theta)$





temperature $T = 1/R$, density of quasiparticles ρ_0

occupation ratio $n = \frac{\rho_0}{\rho_t} = (1 + e^{R\epsilon_0})$

Free energy: $F(T) = -LR \int p'(u) \log(1 + e^{-R\epsilon_0(u)})$

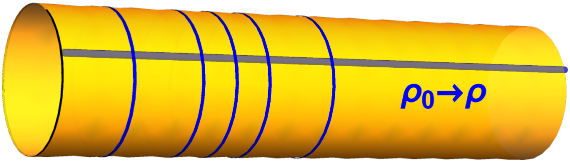
charges $Q_\lambda = \int d\theta \rho(\theta) h_\lambda(\theta)$

2d thermodynamics

$$\text{TBA} \qquad \epsilon_0(\theta) = E(\theta) - \frac{1}{R}\int đu\, \Phi'(\theta - u) \log(1 + e^{-R\epsilon_0(u)})$$

$$\text{BYE} \qquad \rho_t(\theta) = p'(\theta) + \int đu\;\Phi'(\theta - u)\rho_0(u)$$

Due to QFT $\rightarrow$ virtual quasiparticles on R circle are present.



$$\epsilon_0(\theta) = E(\theta) - \frac{1}{R}\int \bar{d}u\, \Phi'(\theta - u) \log(1 + e^{-R\epsilon_0(u)})$$

$$\downarrow$$

$$\epsilon(\theta) = E(\theta) + \frac{i}{R}\sum_j \Phi(\theta - \theta_j^+) - \frac{1}{R}\int \bar{d}u\, \Phi'(\theta - u) \log(1 + e^{-R\epsilon(u)})$$

$$\epsilon(\theta) = E(\theta) + \frac{i}{R}\sum_j \Phi(\theta - \theta_j^+) - \frac{1}{R}\int đu\, \Phi'(\theta - u)\log(1 + e^{-R\epsilon(u)})$$

$$\theta_j^+ = \theta + i\frac{\pi}{2}, \qquad g^\pm(u) = g(u \pm i\tfrac{\pi}{2})$$

This yields BE corrected by the ground state contribution.

$$R\,\epsilon(\theta_i^+) = i(2n_i + 1)\pi$$

$$2n_i\pi = R\,E(\theta_i^+) + \sum_j \Phi(\theta_i - \theta_j) + i\int đu\, \Phi'(\theta_i^+ - u)\log(1 + e^{-R\epsilon(u)})$$

Assume dominant configuration of virtual particles is given by $\tilde{\rho}$

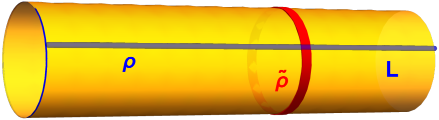
$$\rightarrow \quad \sum_j (.) \longrightarrow \; R \int \bar{d} u \, (.) \, \tilde{\rho}(u)$$

$$\epsilon(\theta) = E(\theta) + i \int du \, \Phi(\theta - u^+) \tilde{\rho}(u) - \frac{1}{R} \int \bar{d} u \, \Phi'(\theta - u) \log(1 + e^{-R\epsilon(u)})$$

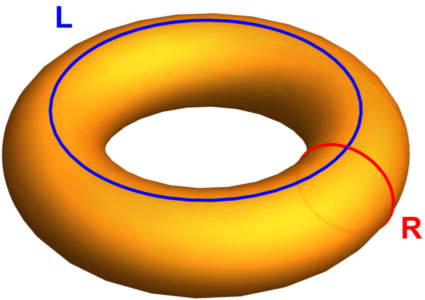
$$\epsilon'(\theta^+) = 2\pi i \tilde{\rho}_t(\theta),$$

$$\tilde{\rho}_t(\theta) = E(\theta) + \int \bar{d} u \, \Phi'(\theta - u) \tilde{\rho}(u) + \frac{i}{R} \int \bar{d} u \, \Phi''(\theta^+ - u) \log(1 + e^{-R\epsilon(u)})$$

BYE!!!, $\quad \Rightarrow \quad \tilde{\rho} \sim$ density of the virtual states.



Torus



The torus as product of two circles:
L – space $\times$ R – the temperature circles $\to$ we can dualize equations:

$$L \leftrightarrow R,\ \rho \leftrightarrow \tilde{\rho}$$

Torus

$$\epsilon(\theta) = E(\theta) + i\int đu\ \Phi(\theta - u^+)\tilde{\rho}(u) - \frac{1}{R}\int đu\, \Phi'(\theta - u)\log(1 + e^{-R\epsilon(u)})$$

$$\tilde{\rho}_t(\theta) = E(\theta) + \int đu\, \Phi'(\theta - u)\tilde{\rho}(u) + \frac{i}{R}\int đu\, \Phi''(\theta^+ - u)\log(1 + e^{-R\epsilon(u)})$$

Torus

$$\epsilon(\theta) = E(\theta) + i\int đu\ \Phi(\theta - u^+)\tilde{\rho}(u) - \frac{1}{R}\int đu\,\Phi'(\theta - u)\log(1 + e^{-R\epsilon(u)})$$

$$\tilde{\rho}_t(\theta) = E(\theta) + \int đu\,\Phi'(\theta - u)\tilde{\rho}(u) + \frac{i}{R}\int đu\,\Phi''(\theta^+ - u)\log(1 + e^{-R\epsilon(u)})$$

$\updownarrow$ $L \leftrightarrow R, \quad \rho \leftrightarrow \tilde{\rho}, \quad \epsilon \leftrightarrow \tilde{\epsilon}$ $\updownarrow$

$$\tilde{\epsilon}(\theta) = E(\theta) + i\int đu\ \Phi(\theta - u^+)\rho(u) - \frac{1}{L}\int đu\,\Phi'(\theta - u)\log(1 + e^{-L\tilde{\epsilon}(u)})$$

$$\rho_t(\theta) = E(\theta) + \int đu\ \Phi'(\theta - u)\rho(u) + \frac{i}{L}\int đu\,\Phi''(\theta^+ - u)\log(1 + e^{-L\tilde{\epsilon}(u)})$$

unknown $\tilde{\epsilon},\ \tilde{\rho}_t,\ \epsilon,\ \rho_t$. Twice as many eqs as for $L \to \infty$.

Torus

Sinh-Gordon

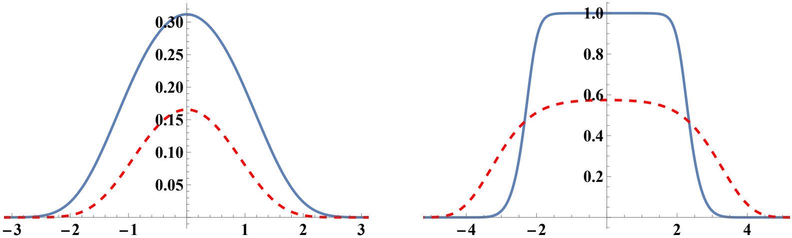
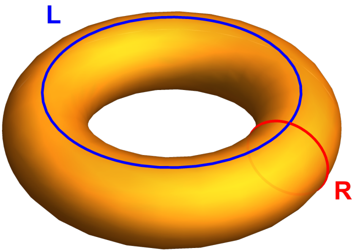


Figure: Plots of $n(\theta)$ (continuum lines) and $\tilde{n}(\theta)$ (dashed lines). The parameters are (a) $Rm = 1,\ Lm = 1.5,\ a = 0.5,\ \mu = 0$ (left figure), and (b) $Rm = 0.5,\ Lm = 0.1,\ a = 0.15,\ \mu = -1.5$ (right figure).

Partition function for free fermion

$$Z = \mathrm{tr}_{\mathcal{H}_{\mathcal{R}}} e^{-LH_R}$$

$$\log Z = -L \int \frac{dp}{2\pi} \log(1 - e^{-R\,E(p)}) - \sum_{n\in\mathbb{Z}} \log(1 - e^{-L\,E(\frac{2\pi n}{R})})$$



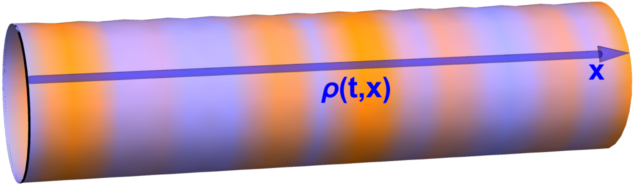
it is symmetric : $(L \leftrightarrow R)$

$$\approx -L \int \frac{dp}{2\pi} \log(1 - e^{-R\,E(p)}) - R \int \frac{dp}{2\pi} \log(1 - e^{-L\,E(p)})$$

$$\uparrow \rho \qquad\qquad \uparrow \tilde{\rho}$$

$L \leftrightarrow R,\ \rho \leftrightarrow \tilde{\rho}$

Generalized hydrodynamics (GHD)



GHD – integrable models – (infinite) current conservation laws.
$\rho(\theta, t, x)$, $n(\theta, t, x)$, etc.,

Charge densities $$q_\lambda(t,x) = \int d\theta\, \rho(\theta, t, x)\, h_\lambda(\theta)$$

$$\partial_t q_\lambda + \partial_x j_\lambda = 0 \qquad j_\lambda = ?$$

$j_\lambda = ?$
Dressing (GHD)

BYE $\quad \rho_t(\theta) = p'(\theta) + \int đu\ \varphi(\theta - u) n(u) \rho_t(u)$

$$\rho_t = p' + \varphi \star n\rho_t, \qquad \rho = n\,\rho_t,\ \varphi = \Phi'$$

$$\downarrow$$

$$\rho_t = (p')^{\mathrm{dr}}$$

dressing $\qquad (g)^{\mathrm{dr}} = g + \varphi \star n\,(g)^{\mathrm{dr}}$

$$q_\lambda(t,x) = \int d\theta h_\lambda n\,(p')^{\mathrm{dr}} = \int d\theta p'\, n\,(h_\lambda)^{\mathrm{dr}}$$

Mirror: $(E, p) \to i(p, E)$, $E \to ip \Rightarrow q_\lambda \to ij_\lambda$

$$q_\lambda(t,x) = \int p' \, n \, (h_\lambda)^{\mathrm{dr}} = \int (p')^{\mathrm{dr}} n \, h_\lambda$$

$$\downarrow$$

$$j_\lambda(t,x) = \int E' n \, (h_\lambda)^{\mathrm{dr}} = \int (E')^{\mathrm{dr}} \, n \, h_\lambda$$

GHD equations.

$$\partial_t q_\lambda + \partial_x j_\lambda = 0$$

$$\downarrow$$

$$(p')^{\mathrm{dr}} \partial_t n + (E')^{\mathrm{dr}} \partial_x n = 0$$

$$(g)^{\mathrm{dr}} = (1 - \varphi \star n) g$$

Generalized hydrodynamics (GHD)

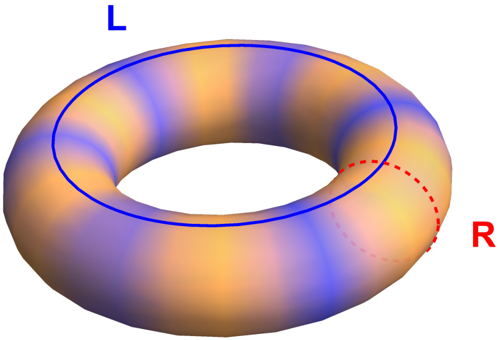


Figure: Non-trivial fluid dynamics (imagine by color shades varying with x) take place along L-circle.

Charges (GHD)

torus TBA $\longrightarrow$ $\rho_t = p' + \varphi \star (n\rho_t + (\varphi^{-1} \star \varphi^-) \star \tilde{n}\rho_t^+)$

the charge density is $n\rho_t + (\varphi^{-1} \star \varphi^-) \star \tilde{n}\rho_t^+$

$$q_\lambda = \int h_\lambda(n\rho_t + (\varphi^{-1} \star \varphi^-) \star \tilde{n}\rho_t^+) = \int (h_\lambda n\rho_t + h_\lambda^+ \tilde{n}\rho_t^+)$$

$$\hat{\rho}_t = \begin{pmatrix} \rho_t \\ \rho_t^+ \end{pmatrix}, \hat{h} = \dots, \hat{n} = \begin{pmatrix} n & 0 \\ 0 & \tilde{n} \end{pmatrix}, \quad g^\pm(u) = g(u \pm i\frac{\pi}{2})$$

charges $\quad q_\lambda = \int \hat{h}_\lambda \hat{n}\, \hat{\rho}_t$

Dressing (tGHD)

$$\hat{\boldsymbol{\rho}}_t = \begin{pmatrix} \boldsymbol{\rho}_t \\ \boldsymbol{\rho}_t^+ \end{pmatrix}, \hat{\varphi} = \begin{pmatrix} \varphi & \varphi^- \\ \varphi^+ & \varphi \end{pmatrix}, \ \hat{n} = \begin{pmatrix} n & 0 \\ 0 & \tilde{n} \end{pmatrix}$$

torus TBA $\longrightarrow \quad \hat{\boldsymbol{\rho}}_t = \hat{p}' + \hat{\varphi} \star \hat{n} \hat{\boldsymbol{\rho}}_t,$

New dressing : $\qquad \widehat{\mathrm{dr}}\,(\hat{g}) = (1 - \hat{\varphi} \star \hat{n})^{-1} \hat{g}$

New dressing $\widehat{\mathrm{dr}}$

$$\hat{\boldsymbol{\rho}}_t = \widehat{\mathrm{dr}}\,(\hat{p}') \longrightarrow = \int \hat{p}' \hat{n}\, \widehat{\mathrm{dr}}\,(\hat{h}_\lambda)$$

$$\hat{1}\cdot\widehat{\mathrm{dr}}^{\,*}(\partial_t\hat{n}\,\widehat{\mathrm{dr}}\,(\hat{p}')+\partial_x\hat{n}\,\widehat{\mathrm{dr}}\,(\hat{E}'))=0.$$

$$\hat{1}=(1,\varphi^{-1}\star\varphi^{-}\star),\ \widehat{\mathrm{dr}}=(1-\hat{\varphi}\hat{n})^{-1},\ \widehat{\mathrm{dr}}^{\,*}=(1-\hat{n}\hat{\varphi})^{-1}.$$

we need to find $\tilde{n}=\tilde{n}(\theta,n(\theta,t,x))=(1+e^{L\tilde{\epsilon}})^{-1}$ from TBA

$$\begin{aligned}\tilde{\epsilon}&=E+i\Phi^{-}\star n\rho_t-\frac{1}{L}\varphi\star\log(1+e^{-L\tilde{\epsilon}})\\ \rho_t&=E+\varphi\star n\rho_t-\frac{i}{L}\varphi'^{-}\star\log(1+e^{-L\tilde{\epsilon}})\end{aligned}$$

$\tilde{n} \ll 1$ approximation

$$\tilde{n} \approx \exp\{-L(E + i\Phi^{-}\star n\,(E)^{\mathrm{dr}})\}$$

$$\partial_t n + \left[\frac{(E')^{\mathrm{dr}} - i\varphi^{+}\star(\tilde{n}\,E)}{(p')^{\mathrm{dr}} - i\varphi^{+}\star(\tilde{n}\,p)} - 2\pi\,L\,(\varphi^{-1}\star\varphi^{-})\star\tilde{n}(E\Phi^{+} - p\Phi^{+}\frac{p}{E}))\star\right]\partial_x n = 0$$